ABSTRACT. The generalized hypergeometric function was introduced by Srivastava and Daoust. In the present paper a new integral representation is derived. Similarly new integral representations of Lauricella and Appell function are obtained.

KEY WORDS. Lauricella, Appell, integral representations, Mellin transformation.

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1. INTRODUCTION.

The generalized Lauricella function of several complex variables was introduced by Srivastava and Daoust [7,8]. The only integral representation which seems to be known for this function is in terms of a Mellin-Barnes integral [6,8]. From a fundamental result about Mellin transforms in n-dimensions, we obtain a new representation for the generalized Lauricella function under suitable restrictions on the parameters. In a similar manner new integral representations are obtained for the Lauricella functions F_{A}^{(n)}, F_{C}^{(n)} and F_{D}^{(n)}, and consequently for the Appell functions F_{1}, F_{2}, and F_{4}. From these derivations it is clear that the method does not provide representations for F_{B}^{(n)} and F_{3}.

2. THE FUNDAMENTAL THEOREM.

Although the following theorem is quite simple, nevertheless it has basic importance.

THEOREM. Let \( f(z)z^{t-1} \in L(0, \infty) \) and

\[
M_{n}[f(z)](s) = \int_{0}^{\infty} f(z)z^{t-1}dz = f^{n}(s).
\]

If \( \text{Re}s_{j} > 0, j = 1, 2, \cdots, n \) and \( \text{Re}\left(\sum_{j=1}^{n}s_{j}\right) = \text{Re}s \), then

\[
M_{n}[f(\max\{x_{1}, \cdots, x_{n}\})](s_{1}, \cdots, s_{n}) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} f(\max\{x_{1}, \cdots, x_{n}\}) \prod_{i=1}^{n} x_{i}^{s_{i}-1}dx_{1} \cdots dx_{n} \]

\[
= \frac{1}{s_{1} \cdots s_{n}} f^{n}(s_{1} + \cdots + s_{n}).
\]

Here \( M_{n} \) represents the n-dimensional Mellin integral transformation [8].

PROOF. In fact we have

\[
\int_{0}^{\infty} \cdots \int_{0}^{\infty} f(\max\{x_{1}, s, x_{n}\}) \prod_{i=1}^{n} x_{i}^{s_{i}-1}dx_{1} \cdots dx_{n} = \\
= \sum_{i=1}^{n} \int_{0}^{\infty} f(x_{i}) \int_{0}^{x_{i}} (n-1) \int_{0}^{x_{j}} \cdots \int_{0}^{x_{j}} x_{j}^{s_{j}-1}dx_{1} \cdots dx_{n} = 
\]
Thus the formula (2) is proved.

The only related results which we have found in the literature are those for the two-dimensional Laplace transformation in the tables of Voelker and Doetsch [9; p. 165, (30),(32)] and in the work of Černov [4; p. 145]. In an analogous manner to our theorem, those results easily can be derived and extended to higher dimensions.

3. THE INTEGRAL REPRESENTATION OF THE GENERALIZED LAURICELLA FUNCTION.

The generalized hypergeometric function of Srivastava and Daoust [6] is defined by

$$ F_{p, q}^{\rho, \sigma} \left( \frac{a_0; c_1, \ldots, c_n}{d_0; b_1, \ldots, b_n}; z_1, \ldots, z_n \right) = \sum_{m_1, \ldots, n_m=0}^{\infty} \prod_{j=1}^{n} \frac{\Gamma(a_j + s_1 + \cdots + s_n)}{\Gamma(b_j + s_1 + \cdots + s_n)} \prod_{i=1}^{n} \left( \frac{\Gamma(\sum_{j=1}^{n} (c_{ij} + m_i)(2m_i + 1))}{\Gamma(\sum_{j=1}^{n} (d_{ij} + 1)m_i)} \right) $$

where for absolute convergence it is sufficient that

$$ 1 + q + q_0 - p - p_k \geq 0; \quad k = 1, 2, \ldots, n. $$

The function in (3) is a special case of the H-functions of several variables which were defined in [3,6]. The fundamental theorem leads to the following result which involves the G-function of Meijer [8]. Let

$$ f(z) = C_{p, q}^{\rho, \sigma} \left( z ^ {i_1, \ldots, i_n} \right) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \Gamma(a_j + s) \Gamma(b_j + s) z^{-s} ds $$

where Re $a_j > 0$, $j = 1, 2, \ldots, p$, and either $p = q$ and Re $\sum_{j=1}^{p} a_j < \sum_{j=1}^{q} b_j$ or $q < p$. It is known [5] that

$$ f^*(s) = \frac{\prod_{j=1}^{p} \Gamma(a_j + s)}{\prod_{j=1}^{q} \Gamma(b_j + s)}, \quad \text{Re}s > 0. $$

Therefore (2) becomes

$$ \int_{0}^{\infty} \cdots \int_{0}^{\infty} C_{p, q}^{\rho, \sigma} \left( \max\{x_1, \ldots, x_n\} \right) \prod_{i=1}^{n} x_i^{s_i - 1} dx_1 \cdots dx_n = $$

$$ = \frac{\prod_{j=1}^{p} \Gamma(a_j + s_1 + \cdots + s_n)}{\prod_{j=1}^{q} \Gamma(b_j + s_1 + \cdots + s_n)} \frac{s_1 + \cdots + s_n}{s_1 \cdots s_n}, \quad \text{Re}s_1, \ldots, \text{Re}s_n > 0. $$

The representation for the generalized Lauricella function can now be obtained as follows. Let $p_i \leq q_i$, $i = 1, 2, \ldots, n$, and either $p = q$ and Re $\sum_{j=1}^{p} a_j < \sum_{j=1}^{q} b_j$ or $p > q$. Then

$$ F_{p, q}^{\rho, \sigma} \left( \frac{a_0; c_1^1, \ldots, c_n^1}{d_0^1; b_1^1, \ldots, b_n^1}; z_1^1, \ldots, z_n^1 \right) = $$

$$ = \frac{1}{n} \sum_{m_1, \ldots, n_m=0}^{\infty} \prod_{j=1}^{n} \frac{\Gamma(b_j_1 + \cdots + b_j_n)}{\Gamma(a_j + b_j_1 + \cdots + b_j_n)(m_1 + \cdots + m_n)n_1} \prod_{i=1}^{n} \left( \prod_{j=1}^{p} \frac{\Gamma(c_{ij} + m_i)(2m_i + 1)}{\Gamma(d_{ij} + 1)m_i} \right) \prod_{i=1}^{n} \left( \prod_{j=1}^{p} \frac{\Gamma(\sum_{j=1}^{n} (c_{ij} + m_i)(2m_i + 1))}{\Gamma(\sum_{j=1}^{n} (d_{ij} + 1)m_i)} \right) $$

$$ \int_{0}^{\infty} \cdots \int_{0}^{\infty} C_{p, q}^{\rho, \sigma} \left( \max\{x_1, \ldots, x_n\} \right) \prod_{i=1}^{n} x_i^{s_i - 1} dx_1 \cdots dx_n. $$
Thus we have proved

\[ F_{\text{p} \text{r} \text{d} \text{t} \text{r} \text{s} \text{r} \text{r} \text{r}}(a; b; c_1, \ldots, c_n; d_1, \ldots, d_n; x_1, \ldots, x_n) = \]

\[ = \frac{1}{n!} \frac{\Gamma(b_1) \cdots \Gamma(b_k)}{\Gamma(a_1) \cdots \Gamma(a_k)} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left( \max\{t_1, \ldots, t_n\}\right)^{b_1-1} \cdots \left( \max\{t_1, \ldots, t_n\}\right)^{b_k-1} \frac{1}{t_1 \cdots t_n} dt_1 \cdots dt_n. \]

Formula (5) is valid when \( \Re a_i > n; i = 1, 2, \ldots; p; p_i \leq q_i, i = 1, 2, \ldots, n; p \geq q, 1 + q + q_i - p_i \geq 0; i = 1, 2, \ldots, n. \) 
(The restriction \( \Re \sum_{j=1}^p b_j + 1 > \Re \sum_{j=1}^p a_j \) is needed when \( p = q. \))

4. INTEGRAL REPRESENTATIONS OF LAURICELLA HYPERGEOMETRIC FUNCTIONS.

Derivations similar to those in the previous section lead to representations for 3 of the Lauricella functions.

(Since in (5) \( p \geq q \) we do not get \( F^{(n)}_B \).) We introduce the operator

\[ D_i = \left( \frac{a - 1}{n} + z_i \frac{d}{dz_i} \right). \]

(a) Using the formula (2) we get

\[ \int_0^\infty \cdots \int_0^\infty \exp(-\max\{z_1, \ldots, z_n\}) \prod_{i=1}^n \frac{z_i^{x_i+1} \cdots z_i^{x^1+1} \cdots z_i^{x^n+1}}{\prod_{i=1}^n (z_i)_{x_i+1} \cdots (z_i)_{x^n+1}} dx_1 \cdots dx_n = \]

\[ = \frac{\Gamma(a + m_1 + \cdots + m_n)}{(\frac{x_1}{x_1} + m_1) \cdots (\frac{x_n}{x_n} + m_n)}, \quad \Re a > 1. \]

Consequently, for the Lauricella function \( F^{(n)}_A \) we have

\[ F^{(n)}_A(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n) = \]

\[ = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1+\cdots+m_n} \cdots (b_n)_{m_1+\cdots+m_n}}{(c_1)_{m_1+\cdots+m_n} \cdots (c_n)_{m_1+\cdots+m_n}} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \frac{z_i^{x_i+1} \cdots z_i^{x^1+1} \cdots z_i^{x^n+1}}{(z_i)_{x_i+1} \cdots (z_i)_{x^n+1}} dt_1 \cdots dt_n = \]

\[ = \frac{\Gamma(a)}{\Gamma(a + m_1 + \cdots + m_n)} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \frac{z_i^{x_i+1} \cdots z_i^{x^1+1} \cdots z_i^{x^n+1}}{(z_i)_{x_i+1} \cdots (z_i)_{x^n+1}} dt_1 \cdots dt_n. \]

Consequently

\[ F^{(n)}_A(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n) = \]

\[ = \frac{1}{\Gamma(a)} \prod_{i=1}^n D_i \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \frac{z_i^{x_i+1} \cdots z_i^{x^1+1} \cdots z_i^{x^n+1}}{(z_i)_{x_i+1} \cdots (z_i)_{x^n+1}} dt_1 \cdots dt_n, \quad \Re a > 1. \]

(b) From (2) we have

\[ \int_0^1 \int_0^1 (1 - \max\{x_1, \ldots, x_n\})^{a-1} \prod_{i=1}^n \frac{z_i^{x_i+1} \cdots z_i^{x^1+1} \cdots z_i^{x^n+1}}{(z_i)_{x_i+1} \cdots (z_i)_{x^n+1}} dx_1 \cdots dx_n = \]

\[ = \frac{\Gamma(a + m_1 + \cdots + m_n)}{\Gamma(b + m_1 + \cdots + m_n)} \frac{\Gamma(b - a + 1)}{(\frac{x_1}{x_1} + m_1) \cdots (\frac{x_n}{x_n} + m_n)} \cdot \quad 1 + \Re a > 1. \]
Therefore for the Lauricella function $F^{(n)}_D$ we get

$$F^{(n)}_D(a; b_1, \ldots, b_n; c; x, \ldots, x_n) =$$

$$= \sum_{m_s = 0}^{\infty} \frac{(a)_{m_s+1} + m_s}{(c)_{m_s+1} + m_s} \frac{(b_1)_{m_s+1}}{1 \cdot \cdots \cdot 1} \frac{\prod_{i=1}^{n} t_i^{m_i} \prod_{i=1}^{n} (1 - \max \{t_i, \ldots, t_n\})^{c-a} \prod_{i=1}^{n} \Gamma(c_m)\Gamma(c-a+1) dt_1 \cdots dt_n}{\Gamma(c)\Gamma(c-a+1) dt_1 \cdots dt_n} \prod_{i=1}^{n} \frac{z_i^{m_i}}{m_i!}$$

Hence

$$F^{(n)}_D(a; b_1, \ldots, b_n; c; x, \ldots, x_n)$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a+1)} \prod_{i=1}^{n} \frac{1}{D_i} \int_0^1 \frac{1}{\prod_{i=1}^{n} t_i^{m_i} \prod_{i=1}^{n} (1 - \max \{t_i, \ldots, t_n\})^{c-a} \prod_{i=1}^{n} (1 - x_i)^{-b_i} dt_1 \cdots dt_n} \prod_{i=1}^{n} \frac{1}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$

(c) Using the formula (2) again, we have

$$\int_0^x \cdots \int_0^x \prod_{i=1}^{n} (\max \{x_1, \ldots, x_n\}) \frac{1}{(a)_{m_i+1}} K_{x-a+1} \left(2(\max \{t_1, \ldots, t_n\})^{x-a} \prod_{i=1}^{n} (x_i)^{m_i+1} \right) \prod_{i=1}^{n} \frac{1}{x_i} \frac{x_i}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$

$$\prod_{i=1}^{n} \frac{1}{x_i} \frac{x_i}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$

Consequently, for the Lauricella hypergeometric function $F^{(n)}_C$, by a similar development we have the result

$$F^{(n)}_C(a; b_1, \ldots, b_n; c; x, \ldots, x_n)$$

$$= \frac{2}{\Gamma(a)\Gamma(b)} \prod_{i=1}^{n} \frac{1}{D_i} \int_0^1 \frac{1}{\prod_{i=1}^{n} t_i^{m_i} \prod_{i=1}^{n} (1 - \max \{t_i, \ldots, t_n\})^{b_i+1} K_{a+b+1} \left(2(\max \{t_1, \ldots, t_n\})^{x-a} \prod_{i=1}^{n} (x_i)^{m_i+1} \right) \prod_{i=1}^{n} \frac{1}{x_i} \frac{x_i}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$

$$\prod_{i=1}^{n} \frac{1}{x_i} \frac{x_i}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$

For Appell functions these read

$$F_1(a; b_1, b_2; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a+1)}$$

$$D_1D_2 \int_0^1 \int_0^1 (1 - \max \{t_1, t_2\})^{c-a} \frac{(1 - x_1)^{a-1}}{(1 - x_2)^{a-1}} \frac{1}{(1 - x_2)^{a-1}} dt_1 dt_2$$

$$\prod_{i=1}^{n} \frac{1}{x_i} \frac{x_i}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$

$$\prod_{i=1}^{n} \frac{1}{x_i} \frac{x_i}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$

For Appell functions these read

$$F_2(a; b_1, b_2; c_1, c_2; x) =$$

$$= \frac{1}{\Gamma(a)D_1D_2} \int_0^1 \int_0^1 (t_1t_2)^{a-3/2} \exp(-\max \{t_1, t_2\})$$

$$\left\{ \frac{1}{(1 - x_1)^{-b_1}} \frac{1}{(1 - x_2)^{-b_2}} \right\} dt_1 dt_2$$

$$\prod_{i=1}^{n} \frac{1}{x_i} \frac{x_i}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$

$$\prod_{i=1}^{n} \frac{1}{x_i} \frac{x_i}{n + m_i} \frac{(b_i)_{m_i}}{m_i!}$$
\begin{equation}
F_{4}(a; b; c_1, c_2; x_1, x_2) = \frac{2\Gamma(c_1)\Gamma(c_2)}{\Gamma(a)\Gamma(b)} D_1 D_2 \int_{0}^{\infty} \int_{0}^{\infty} \left(\max\{t_1, t_2\}\right)^{(a-\alpha+1)/2} \left(\max\{t_1, t_2\}\right)^{(a-\beta+1)/2} K_{\beta-\alpha+1} \left(2\left(\max\{t_1, t_2\}\right)^{1/2}\right)
\left\{\left(x_1 t_1\right)^{(1-c_1)/2} I_{c_1-1} \left(2\left(x_1 t_1\right)^{1/2}\right)\right\} \left\{\left(x_2 t_2\right)^{(1-c_2)/2} I_{c_2-1} \left(2\left(x_2 t_2\right)^{1/2}\right)\right\} dt_1 dt_2,
\end{equation}

\text{Re} a > 1, \quad \text{Re} b > 0.

**References**


