In 1948, Samuel [2] pointed out that the intersection of two group topologies need not be a group topology. However, a number of properties that hold for a group topology still hold for a topological space that is an intersection of group topologies. In order to study these properties, we shall describe a class of topologies that can be placed on a group which we call semicontinuous topologies. (We point out here that Fuchs [1] calls these spaces semitopological groups).

One important attribute of topological groups is separation. In particular, a topological group is Hausdorff if and only if the identity is a closed subset. While this is not true for semicontinuous groups, we shall see that an interesting "echo" of this property is true.

For each group $G$ we have a bijection $\text{inv}: G \to G$ defined by $\text{inv}(x) = x^{-1}$. Also for any fixed $a \in G$ we have bijections $\text{la}: G \to G$ defined by $\text{la}(x) = az$ and $\text{ra}: G \to G$ defined by $\text{ra}(x) = xa$.

DEFINITION. A semicontinuous group is a group $G$ and a topology $\tau$ on $G$ making $\text{inv}$, $\text{la}_a$, and $\text{ra}_a$ continuous for $a \in G$.

Clearly a semicontinuous group is a homogeneous space. Thus a great deal can be determined by considering a basis for the topology at the identity. In a manner analogous to that found in the theory of topological groups, one can demonstrate the following:

PROPOSITION 1. If $(G, \tau)$ is a semicontinuous group and $\mathcal{U}$ is a neighborhood base at the identity, then $\mathcal{U}$ satisfies

(i) If $U, V \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $W \subseteq U \cap U$.
(ii) If $a \in U$ and $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $Va \subseteq U$.
(iii) If $a \in U \in \mathcal{U}$ then there exists $V \in \mathcal{U}$ such that $Va \subseteq U$.
(iv) If $a \in U \in \mathcal{U}$ and $x \in G$ then there exists $V \in \mathcal{U}$ such that $xV^{-1} \subseteq U$.

Furthermore, if $\mathcal{U}$ is any collection of subsets of $G$, each containing the identity, and $\mathcal{U}$ satisfies (i)-(iv) above, then there exists a unique semicontinuous topology $\tau$ on $G$ for which $\mathcal{U}$ is a neighborhood base at the identity.
Any collection of subsets $\mathcal{Y}$ satisfying (i)-(iv) is called a semifundamental system. Let $V = \{x \mid x = r + \sqrt{2} \text{ and } r \in \mathbb{Q}\} \subseteq \mathbb{R}$ and let $W$ be the collection of all translation sets $a + V$ such that $0 \in a + V$. Finally let $\mathcal{Y}$ be the collection of all finite intersections of elements of $W$.

A moment's reflection shows that $\mathcal{Y}$ is a semifundamental system that generates a topology $\tau$ which is finer than the usual topology on $\mathbb{R}$. The set $Q$ is closed in $(\mathbb{R}, \tau)$. Yet the quotient topology generated on $\mathbb{R}/Q$ by projection from $(\mathbb{R}, \tau)$ is the finite complement topology. Therefore the separation properties for semicontinuous groups are clearly different from those found in topological groups.

Another interesting example of a semicontinuous topology can be described as follows; let $B_n$ be the open ball of radius $1/n$ centered at the origin of the plane, and let $V_n = B_n - \{(x, y) \mid 0 < \frac{1}{n}x \leq y \leq nx\}$. The collection of sets $\{V_n\}_{n=2}^{\infty}$ forms a semifundamental system for the group $(\mathbb{R}^2, +)$. The relative topology on $(\mathbb{Q}^2, +)$ is an example of a second countable metric space that cannot be a topological group since no square of an open set can be placed inside $V_n$.

Let $(G, t)$ be a semicontinuous group and $m: G \times G \to G$ the multiplication map. We let $q(t)$ denote the quotient topology on $G$ generated by $m$ when the product topology $t \times t$ is placed on $G \times G$. If $N$ is a normal subgroup of $G$ and $(G, t)$ is a semicontinuous group, we shall denote the quotient topology on $G/N$ generated by the natural map $\pi: G \to G/N$, by $\pi(t)$.

**Lemma 2.** If $(G, t)$ is a semicontinuous group, then both $m$ and $\pi$ are open maps and both $G/N$ and $(G, q(t))$ are semicontinuous groups.

**Proof.** Let $U \times V$ be a basic open set in $t \times t$. Then $m^{-1}(m(U \times V)) = \bigcup_{g \in G} (Ug \times g^{-1}V)$. Therefore $m$ is an open map. Likewise $\pi^{-1}((U \times U)) = UN$ which is open in $(G, t)$ whenever $U \times U$. Thus $\pi$ is an open map.

Since $\lambda \times id: (G \times G, t \times t) \to (G \times G, t \times t)$ is continuous and $q(t)$ is a quotient topology, $m_\lambda(G, q(t)) \to (G, q(t))$ is continuous. Similar arguments show that the maps $r_{\lambda_\lambda}: (G, q(t)) \to (G, q(t))$ and $inv: (G, q(t)) \to (G, q(t))$ are continuous. The proof that the quotient topology on $G/N$ is semicontinuous is done in the same fashion.

**Lemma 3.** If $S \subseteq G$ then $S = \bigcap_{V \in \mathcal{Y}} VS$.

**Proof.** $x \notin \bigcap_{V \in \mathcal{Y}} V$ iff there exists $W \in \mathcal{Y}$ with $x \notin WS$ iff $W^{-1}x \cap S = \emptyset$.

**Theorem 4.** $G/N$ is Hausdorff iff $N = \bigcap_{V \in \mathcal{Y}} V^2$.

**Proof.** We consider the following commutative diagram:

$$
\begin{array}{ccc}
G \times G & \xrightarrow{\pi} & G/N \\
\downarrow m & & \downarrow \pi \\
G & \xrightarrow{\pi} & G/N
\end{array}
$$

We have that $\{V^2 \mid V \in \mathcal{Y}\}$ is a semifundamental system for $q(t)$ whenever $\mathcal{Y}$ is a semifundamental system for $t$. The identity element in $(G/N, \pi(q(t)))$ will be closed if and only if $N = \bigcap_{V \in \mathcal{Y}} V^2$. The identity element in $(G/N, q(\pi(t)))$ will be closed if and only if the diagonal is closed in $G/N \times G/N$. However $\pi(q(t)) = q(\pi(t))$ since the maps are open.

**Corollary 5.** $(G, t)$ is Hausdorff if and only if $\bigcap_{V \in \mathcal{Y}} V^2 = \{e\}$. 

COROLLARY 6. If \((G,t)\) is a minimal Hausdorff semicontinuous group then \((G,t)\) is topological group if and only if \(\bigcap_{V \in \mathcal{V}} V^4 = \{e\}\).

We can define an equivalence relation on \((G,t)\) by defining \(x \sim y\) if and only if there does not exist \(V \in \mathcal{V}\) such that \(xV \cap yV = \phi\). Let \(K\) denote the equivalence class of \(e\) under this equivalence relation. We call \(K\) the **Hausdorff Kernel** of \((G,t)\).

**THEOREM 7.** \(K = \bigcap_{V \in \mathcal{V}} V^2\) and \(K\) is the minimum normal subgroup with the property that \(G/K\) is Hausdorff.

**PROOF.** We note by Lemma 3 that \(\bigcap_{V \in \mathcal{V}} V^2\) is the closure of \(\{e\}\) in \((G,q(t))\). Therefore by an argument similar to that for topological groups, \(\bigcap_{V \in \mathcal{V}} V^2\) is a normal subgroup of \(G\). Since we can without loss of generality assume that \(V\) is symmetric, it is clear the \(K = \bigcap_{V \in \mathcal{V}} V^2\). The proof of Theorem 4 shows that \(G/K\) is Hausdorff if and only if \(K\) is closed in \((G,q(t))\). But \(K\) is the smallest closed normal subgroup in \((G,q(t))\).

In a like manner we can define an equivalence relation on \((G,t)\) by declaring \(x \sim y\) if and only if there does not exist a continuous function \(\phi: G \to R\) with \(\phi(x) \neq \phi(y)\). The equivalence class of \(e\) under this relation will also be a closed normal subgroup that we call the **completely Hausdorff kernel** of \((G,t)\).

**REFERENCES**

