A METHOD OF SOLVING $y^{(k)} - f(x)y = 0$

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**ABSTRACT.** An alternative method is shown for solving the differential equation $y^{(k)} - f(x)y = 0$ by means of series. Also included is a result for a sequence of functions $\{S_n(x)\}$ which gives conditions under which $\lim_{n \to \infty} \left( \frac{d^n}{dx^n} S_n(x) \right) = \frac{d^n}{dx^n} \left( \lim S_n(x) \right)$.

**KEY WORDS AND PHRASES.** Differential equations, sequence of functions.

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1. Introduction

Consider the differential equation $y^{(k)} - f(x)y = 0$ for $a < x < b$ where $f$ is a given function continuous on $a \leq x \leq b$. If $f$ is analytic then the method of power series may be used to solve for $y$. However, for more general $f$, heuristics suggest that one "iterate" to a solution by finding a sequence of functions $\{f_n(x)\}$ that satisfy $f_n^{(k)}(x) = f_{n-1}(x)f(x)$. Then possibly $\sum_{n=1}^{\infty} f_n(x)$ is a solution. Under suitable hypothesis this is indeed the case. The results can be generalized to the differential equation $y^{(k)} - f(x)y = 0$ as shown in Theorem A. The proof depends on an interesting result, Theorem 1, which gives conditions that insure that the limit of the $k$th derivative is the $k$th derivative of the limit. Theorem 1 generalizes the usual result found in Advanced Calculus books for differentiating the limit of a sequence of functions. We also include two examples that illustrate the method of solution when $k = 2$.

2. Statement of Theorems

**Theorem A.** Suppose $f$ is continuous on $[a, b]$, $c \in [a, b]$, and $k$ is a natural number. Define the sequence of functions $\{f_n(x)\}$ by

$$f_n(x) = a_0^{(n)} + a_1^{(n)}x + \ldots + a_{k-1}^{(n)}x^{k-1},$$

where $a_0^{(n)}, a_1^{(n)}, \ldots, a_{k-1}^{(n)}$ are constants, $n = 0, 1, 2, \ldots$. (Note $f_n(x)$ is any $k$th antiderivative of $f_n(x)f(x)$.)

If the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on $[a, b]$ to some function $S(x)$ then $\sum_{n=0}^{\infty} f_n^{(k)}(x)$ converges uniformly to $S^{(k)}(x)$ for $a \leq x \leq b$, $j = 1, 2, \ldots, k$, and $S^{(k)}(x) = S(x) \cdot f(x)$ on $[a, b]$. 


Remark: All derivatives at the endpoints $a$ and $b$ are necessarily one sided.

As mentioned, the proof of Theorem A depends on the following interesting result for sequences of differentiable functions:

**Theorem 1.** Suppose (i) \( \{S_n(x)\}_{n=1}^\infty \) is a sequence of real functions defined on an interval \([a,b]\) and $k$ is a natural number;

(ii) $S_n'(x), S_n''(x), \ldots, S_n^{(k)}(x)$ exist at each $x \in [a,b], n = 1, 2, \ldots$

(iii) \( \{S_n^{(k)}(x)\}_{n=1}^\infty \) converges uniformly on \([a,b]\);

(iv) either there is a $c \in [a,b]$ such that each of \( \{S_n(c)\}_{n=1}^\infty \), \( \{S_n'(c)\}_{n=1}^\infty \), \( \{S_n''(c)\}_{n=1}^\infty \), \ldots, \( \{S_n^{(k-1)}(c)\}_{n=1}^\infty \) converge or there are distinct points $c_1, \ldots, c_l$ such that each of \( \{S_n(c_1)\}_{n=1}^\infty \), \( \{S_n(c_2)\}_{n=1}^\infty \), \ldots, \( \{S_n(c_l)\}_{n=1}^\infty \) converge.

Then each of the sequences \( \{S_n^{(j)}(x)\}_{n=1}^\infty \) converges uniformly on \([a,b]\) to differentiable functions, $j = 0, 1, 2, \ldots, k-1$, and

$$
\frac{d^j}{dx^j} \left( \lim_{n \to \infty} S_n(x) \right) = \lim_{n \to \infty} \left( \frac{d^j}{dx^j} S_n(x) \right), \quad j = 1, 2, \ldots, k.
$$

3. Discussion and Proofs

In order to prove Theorem 1 we need some preliminary results. First is a standard result from Advanced Calculus.

**Theorem 0.** Suppose that \( \{S_n(x)\}_{n=1}^\infty \) is a sequence of real functions differentiable on an interval $a \leq x \leq b$ and such that

(i) \( \{S_n'(x)\}_{n=1}^\infty \) converges uniformly on \([a,b]\);

(ii) \( \{S_n(c)\}_{n=1}^\infty \) converges for some $c \in [a,b]$.

Then \( \{S_n(x)\}_{n=1}^\infty \) converges uniformly on \([a,b]\) to a function $S(x)$, and

$$
\lim_{n \to \infty} \left( \frac{d}{dx} S_n(x) \right) = S'(x) = \lim_{n \to \infty} \left( \frac{d}{dx} S_n(x) \right), \quad a \leq x \leq b.
$$

For a justification of Theorem 0, see [1], pp. 451-2.

Also required is the following

**Lemma.** Suppose $k$ is a natural number, \( \{p_n(x)\}_{n=1}^\infty \) is a sequence of polynomials each of degree $\leq k$, and $c_1, \ldots, c_{k+1}$ are $k+1$ distinct numbers. If \( \{p_n(c_j)\}_{n=1}^\infty \) converges for $j = 1, \ldots, k+1$ then \( \{p_n(x)\}_{n=1}^\infty \) converges for each $x \in \mathbb{R}$ to a polynomial $h(x)$ where either $h(x) = 0$ or degree of $h(x)$ is $\leq k$, and convergence is uniform on each bounded closed interval in $\mathbb{R}$. Moreover, \( \lim_{n \to \infty} p_n^{(v)}(x) = h^{(v)}(x) \), $v = 1, \ldots, k, x \in \mathbb{R}$.

**Proof:** Let $Q(x) = (x-c_1)(x-c_2)\ldots(x-c_{k+1})$. Using the Lagrange Interpolation formula, we have for each $n$,

$$
p_n(x) = \sum_{j=1}^{k+1} \frac{p_n(c_j)Q(x)}{Q'(c_j)(x-c_j)}.
$$
Clearly for each \( x \),
\[
h(x) := \lim_{n \to \infty} p_n(x) = \sum_{j=1}^{k+1} \left[ \lim_{n \to \infty} p_n(c_j) \right] Q(x) / Q'(c_j)(x-c_j)
\]
exists and is finite; moreover \( h(x) \) is a polynomial of degree \( \leq k \). For each \( x \) in some interval \([a, \beta]\),
\[
|p_n(x) - h(x)| \leq \sum_{j=1}^{k+1} \left| \frac{p_n(c_j) - h(c_j)}{Q'(c_j)(x-c_j)} \right| \leq M \sum_{j=1}^{k+1} |p_n(c_j) - h(c_j)|
\] (2.2)
where \( M > 0 \) is such that
\[
\max_{x \in [a, \beta]} \left| \frac{Q(x)}{Q'(c_j)(x-c_j)} \right| \leq M \quad \text{for} \quad j = 1, 2, \ldots, k + 1.
\]
Uniform convergence follows from inequality (2.2). By first differentiating (2.1) and then passing to the limit with \( n \) we obtain
\[
\lim_{n \to \infty} p_n^{(v)}(x) = h^{(v)}(x), \quad x \in \mathbb{R}.
\]

We proceed to the

**Proof of Theorem 1.** We use induction on \( k \). The case \( k = 1 \) is given by Theorem 0. So assume the theorem holds for \( k - 1 \geq 1 \) and let \( \{S_n(x)\}_{n=1}^{\infty} \) satisfy (i)-(iv). If \( \{S_n(c_j)\}_{n=1}^{\infty}, \{S_n'(c_j)\}_{n=1}^{\infty}, \ldots, \{S_n^{(k-1)}(c_j)\}_{n=1}^{\infty} \) each converge then \( \{S_n^{(k-1)}(x)\}_{n=1}^{\infty} \) converges uniformly by Theorem 0 and hence the conclusion follows from the induction hypothesis and Theorem 0. Next suppose \( \{S_n(c_j)\}_{n=1}^{\infty}, \{S_n'(c_j)\}_{n=1}^{\infty}, \ldots, \{S_n^{(k)}(c_j)\}_{n=1}^{\infty} \) each converge and define
\[
G_{n,1}(x) = \int_{c_1}^{x} S_n^{(1)}(u)du - S_n^{(1)}(x) = S_n^{(1)}(x) - S_n^{(1)}(c_1)
\]
\[
G_{n,2}(x) = \int_{c_2}^{x} G_{n,1}(u)du - S_n^{(2)}(x) = S_n^{(2)}(x) - S_n^{(2)}(c_2)
\]
\[
\ldots
\]
\[
G_{n,k}(x) = \int_{c_k}^{x} G_{n,k-1}(u)du - S_n^{(k)}(x) = S_n^{(k)}(x) - p_n(x)
\]
where \( p_n(x) := S_n(x) - \int_{c_1}^{x} \int_{c_2}^{x} \ldots \int_{c_k}^{x} S_n^{(1)}(u_1)du_2 \ldots du_k = S_n(x) - G_{n,k}(x) \) is a uniquely determined polynomial of degree \( \leq k - 1 \), each \( n \). Repeated use of Theorem 0 shows \( \{G_{n,j}(x)\}_{n=1}^{\infty} \) converges uniformly on \([a, \beta]\), for \( j = 1, 2, \ldots, k \), and in particular \( \{G_{n,k}(x)\}_{n=1}^{\infty} \) converges uniformly. Since \( \{S_n(c_j)\}_{n=1}^{\infty} \) converges then \( p_n(c_j) \) \( \to \) \( c_j \) \( \to \) \( \beta \) by the lemma the sequence of polynomials \( \{p_n(x)\}_{n=1}^{\infty} \) converges uniformly to a polynomial \( h(x) \) where either \( h(x) = 0 \) or degree of \( h(x) \) is \( \leq k - 1 \), and \( \{p_n^{(k-1)}(x)\}_{n=1}^{\infty} \) converges to \( h^{(k-1)}(x) \). Because \( p_n^{(k-1)}(x) = S_n^{(k-1)}(x) \), \( n = 1, 2, \ldots \), then \( \{S_n^{(k-1)}(c_j)\}_{n=1}^{\infty} \) converges. It follows that \( \{S_n^{(k-1)}(x)\}_{n=1}^{\infty} \) converges uniformly. Also \( \lim_{n \to \infty} S_n^{(1)}(x) = \frac{d}{dx} \left( \lim_{n \to \infty} S_n^{(1)}(x) \right) \). Now the induction hypothesis can be used and the conclusion obtained.
We are now able to give the

**Proof of Theorem A.** Apply Theorem 1 to the sequence of real functions

\[ S_n(x) = f_0(x) + f_1(x) + \ldots + f_n(x), \quad n = 0, 1, 2, \ldots. \]

Note that each \( S_n(x) \) has \( k \) derivatives and

\[ S^{(k)}_n(x) = (f_0(x) + f_1(x) + \ldots + f_{n-1}(x)) \cdot f(x) \]

\[ = S_{n-1}(x) \cdot f(x) \quad \text{for} \quad n \geq 1. \]

By hypothesis \( \{S_n(x)\}_{n=1}^{\infty} \) converges uniformly on \([a, b]\) to \( S(x) \). Hence \( \{S^{(k)}_n(x)\}_{n=1}^{\infty} \) converges uniformly to \( S(x) \cdot f(x) \). By Theorem 1, we obtain that the sequence of functions \( \{S^{(k)}_n(x)\}_{n=1}^{\infty} \) converges uniformly and

\[ \lim_{n \to \infty} \left( \frac{S_n(x)}{f(x)} \right) = \lim_{n \to \infty} \left( \frac{d^n}{dx^n} S_n(x) \right) \quad \text{for} \quad j = 1, 2, \ldots, k. \]

Thus for \( a \leq x \leq b \),

\[ \lim_{n \to \infty} \left( \frac{S_n(x)}{f(x)} \right) = S(x)/f(x). \]

This proves Theorem A. \( \blacksquare \)

4. **Examples and Remarks**

We give some applications of Theorem A.

**Example 1:** Consider \( y'' - (Ax^k)y = 0 \), \( a \leq x \leq b \), where \( A, k \) are constants, \( k \geq 0 \), and \( f(x) = Ax^k \) is continuous and bounded by \( M \) on \([a, b]\). We may assume \( c = 0 \in [a, b] \) and \( |a| \leq |b| \). Let \( f_0(x) = 1 \) and for \( n \geq 1 \) let

\[ f_n(x) = \frac{A^2 x^{2k+4}}{(k+1)(k+2)(2k+3)(2k+4)}, \quad \ldots \]

\[ f_n(x) = \frac{A^k x^{k+2}}{(k+1)(k+2)(2k+3)(2k+4)}, \quad \ldots \]

Thus \( f_n''(x) = f_{n-1}(x) \cdot f(x) \) and \( |f_n(x)| \leq \frac{M^2}{(2k+3)(2k+4)} \) for \( a \leq x \leq b, n = 1, 2, \ldots \). The series \( \sum_{n=0}^{\infty} f_n(x) \) converges by the ratio test so \( \sum_{n=0}^{\infty} f_n(x) \) converges uniformly on \([a, b]\) to a function \( S(x) \) by the Weierstrass M-test. Now let \( g_0(x) = x \) and for \( n \geq 1 \), let

\[ g_n(x) = \frac{A x^{k+3}}{(k+2)(k+3)}, \quad g_n(x) = \frac{A^2 x^{2k+5}}{(k+1)(k+2)(2k+3)(2k+4)}, \quad \ldots \]

\[ g_n(x) = \frac{A^k x^{k+2}}{(k+1)(k+2)(2k+3)(2k+4)}, \quad \ldots \]

As before \( g_n''(x) = g_{n-1}(x) \cdot f(x) \) and \( |g_n(x)| \leq \frac{M^2}{(2k+3)(2k+4)} \) for \( a \leq x \leq b, n = 1, 2, \ldots \) so that \( \sum_{n=0}^{\infty} g_n(x) \) converges uniformly on \([a, b]\) to a function \( T(x) \). By Theorem 1, \( S(x) \) and \( T(x) \) are solutions to \( y'' - Ax^k y = 0 \). Since
the Wronskian of \( S(x) \) and \( T(x) \) is \( W(x) = S(x)T'(x) - T(x)S'(x) \) and \( W(0) = 0 \) then \( S(x) \) and \( T(x) \) are linearly independent on \([a,b]\), see e.g. [2], pp. 111-113. It follows that the general solution is \( C_1 S(x) + C_2 T(x) \) for constants \( C_1, C_2 \). In particular if \( k = 0 \) and \( A > 0 \) then

\[
S(x) = \sum_{n=0}^{\infty} \frac{(\sqrt{A} x)^{2n}}{(2n)!} = \cosh(\sqrt{A} x) \quad \text{and} \quad T(x) = \frac{1}{\sqrt{A}} \sum_{n=0}^{\infty} \frac{(\sqrt{A} x)^{2n+1}}{(2n+1)!} = \frac{\sin(\sqrt{A} x)}{\sqrt{A}}
\]

and if \( k = 0 \) and \( A < 0 \) then

\[
S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{|A|} x)^{2n}}{(2n)!} = \cos(\sqrt{|A|} x) \quad \text{and} \quad T(x) = \frac{1}{\sqrt{|A|}} \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{|A|} x)^{2n+1}}{(2n+1)!} = \frac{\sin(\sqrt{|A|} x)}{\sqrt{|A|}}
\]

These solutions are the same as those obtained by elementary methods.

**Example 2:** Consider \( y'' - Ae^{kx}y = 0, \ -a < x < a \), where \( A, k \) are constants, \( a > 0, k = 0 \) and \( f(x) = Ae^{kx} \).

Let \( f_0(x) = 1, \quad f_1(x) = \frac{A}{k^2} e^{kx}, \quad f_2(x) = -\frac{A^2 e^{2kx}}{(k^2)(2k)^2}, \ldots \)

Then \( f_n''(x) = f_{n-1}(x)f(x) \) and \( f_n(x) \leq \frac{|A|^n |f_0(x)|}{|x|^n} \) for \( |x| \leq a, n = 1,2,\ldots \) The series \( \sum_{n=1}^{\infty} \frac{|A|^n |f_0(x)|}{|x|^n} \) converges by the Ratio Test so \( \sum_{n=1}^{\infty} f_n(x) \) converges uniformly on \([-a,a]\) to a function \( S(x) \). Now let \( c = 0, g_0(x) = x, \)

\[
g_1(x) = \frac{A e^{kx}}{k^2} \left( x - \frac{2}{k} \right), \quad g_2(x) = \frac{A^2 e^{2kx}}{(k^2)(2k)^2} \left( x - \frac{2}{k} \left( 1 + \frac{1}{2} \right) \right), \ldots
\]

Then \( g_n''(x) = g_{n-1}(x) \cdot f(x) \) and \( |g_n(x)| \leq \frac{|A|^n |f_0(x)|}{|x|^n} \left( |a| + \frac{2}{k} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} \right) \right) = b_n \) for \( |x| \leq a, n = 1,2,\ldots \).

Since \( \sum b_n \) converges by the Ratio Test then \( \left( x + \sum_{n=1}^{\infty} g_n(x) \right) \) converges uniformly on \([-a,a]\) to some \( T(x) \).

To see that \( S(x) \) and \( T(x) \) are linearly independent let \( x = x(u) = k^{-1} \ln u \Rightarrow u = e^{kx} \). Then

\[
S(x(u)) = 1 + \sum_{n=1}^{\infty} \frac{A^* u^n}{(k^* + n)^2}
\]

and

\[
T(x(u)) = \frac{\ln u}{k} + \sum_{n=1}^{\infty} \frac{A^* u^n}{(k^* + n)^2} \left[ \frac{\ln u}{k} - \frac{2}{k} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} \right) \right]
\]

Note \( S(x(u)) \) has a Maclaurin Series but \( T(x(u)) \) does not; hence \( S(x) \) and \( T(x) \) are linearly independent. Thus \( C_1 S(x) + C_2 T(x) \), \( C_1, C_2 \) constants, is the general solution to

\[
y'' - Ae^{kx}y = 0, \quad -a \leq x \leq a.
\]
We conclude with some remarks. Theorem 1 is a generalization of Theorem 0 and is of interest by itself. It is possible to improve Theorem 1 by generalizing hypothesis (iv) e.g., to include in (iv) a third alternative as follows: \( c_1, \ldots, c_{k-1} \) are distinct points and each of \( \{S_n(c_1)\}_{1}^{*}, \ldots, \{S_n(c_{k-1})\}_{1}^{*} \) and \( \{S_n'(c_1)\}_{1}^{*} \) converge. It may be possible to generalize Theorem A to differential equations that include intermediate derivatives, e.g., \( y'' + g(x)y' + f(x)y = 0 \), \( f(x) \) and \( g(x) \) continuous on \( [a,b] \); such a generalization would require an improvement of Theorem 1. As seen in the examples, linearly independent solutions to the differential equation are obtained by using linearly independent functions for \( f(x) \). Finally, we note that difficulties in the application of Theorem A may occur when finding the \( k \)th antiderivative of \( f_{n-1}(x)f(x) \), and thus this method of solution may be impractical for such cases.

*Note: The first author is deceased.

References
