ON CERTAIN BAZILEVIĆ FUNCTIONS OF ORDER \( \beta \)

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ABSTRACT. A certain class \( B(n, \alpha, \beta) \) of Bazilević functions of order \( \beta \) in the unit disk is introduced. The object of the present paper is to derive some properties of functions belonging to the class \( B(n, \alpha, \beta) \). Our result for the class \( B(n, \alpha, \beta) \) is the improvement of the theorem by N. E. Cho \([1]\).

KEY WORDS AND PHRASES. Analytic function, class \( B(n, \alpha, \beta) \), Bazilević function.

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1. INTRODUCTION.

Let \( A(n) \) denote the class of functions of the form

\[
f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in N = \{1, 2, 3, \ldots\})
\]

which are analytic in the unit disk \( U = \{z: |z| < 1\} \). A function \( f(z) \in A(n) \) is said to be a member of the class \( B(n, \alpha, \beta) \) if it satisfies

\[
\text{Re} \left\{ \frac{f'(z)f(z)\alpha - 1}{z^{\beta - 1}} \right\} > \beta
\]

for some \( \alpha (\alpha > 0) \), \( \beta (0 \leq \beta < 1) \), and for all \( z \in U \). We note that \( B(n, \alpha, \beta) \) is the subclass of Bazilević functions in the unit disk \( U \) (cf. \([1]\)). Also we say that \( f(z) \) in the class \( B(n, \alpha, \beta) \) is a Bazilević function of order \( \beta \).

Recently, Cho \([1]\) has studied the class \( B(n, \alpha, 0) \) when \( \beta = 0 \), and has proved

THEOREM A. If \( f(z) \in B(n, 2, 0) \) when \( \alpha = 2 \) and \( \beta = 0 \), then

\[
\text{Re} \left\{ \frac{f(z)}{z} \right\} > \frac{n}{n + 2} \quad (z \in U).
\]

In the present paper, we improve the above theorem by Cho \([1]\).

2. PROPERTIES OF THE CLASS \( B(n, \alpha, \beta) \).

In order to establish our main result, we have to recall here the following lemma due to Miller and Mocanu \([2]\).

LEMMA. Let \( \phi(u, v) \) be a complex valued function,

\[
\phi: D \to C, D \subset C^2 \ (C \text{ is the complex plane}),
\]
and let \( u = u_1 + iu_2, \ v = v_1 + iv_2 \). Suppose that the function \( \phi(u, v) \) satisfies

(i) \( \phi(u, v) \) is continuous in \( D \);
(ii) \( (1,0) \in D \) and \( \text{Re}\{\phi(1,0)\} > 0 \);
(iii) for all \((iu_2, v_1) \in D \) such that \( v_1 \leq -n(1 + u_2^2)/2 \),

\[ \text{Re}\{\phi(iu_2, v_1)\} \leq 0. \]

Let \( p(z) = 1 + p_nz^n + p_{n+1}z^{n+1} + \ldots \) be regular in the unit disk \( U \) such that \( (p(z), zp'(z)) \in D \) for all \( z \in U \). If

\[ \text{Re}\{\phi(p(z), zp'(z))\} > 0 \quad (z \in U), \]

then

\[ \text{Re}\{p(z)\} > 0 \quad (z \in U). \]

Using the above lemma, we prove

**THEOREM 1.** If \( f(z) \in B(n, \alpha, \beta) \), then

\[ \text{Re}\left\{\left(\frac{f(z)}{z}\right)^\alpha\right\} > \frac{n + 2\alpha\beta}{n + 2\alpha} \quad (z \in U). \tag{2.1} \]

**PROOF.** We define the function \( p(z) \) by

\[ \left\{\frac{f(z)}{z}\right\} = \gamma + (1 - \gamma)p(z) \tag{2.2} \]

with

\[ \gamma = \frac{n + 2\alpha\beta}{n + 2\alpha}. \tag{2.3} \]

Then, we see that \( p(z) = 1 + p_nz^n + p_{n+1}z^{n+1} + \ldots \) is regular in \( U \).

It follows from (2.2) that

\[ \frac{f'(z)f(z)\gamma - 1}{z\gamma - 1} = \gamma + (1 - \gamma)p(z) + \frac{(1 - \gamma)zp'(z)}{\alpha}, \tag{2.4} \]

or

\[ \text{Re}\left\{\frac{f'(z)f(z)\gamma - 1}{z\gamma - 1} - \beta\right\} \tag{2.5} \]

\[ = \text{Re}\left\{\gamma - \beta + (1 - \gamma)p(z) + \frac{(1 - \gamma)zp'(z)}{\alpha}\right\} \]

\[ > 0. \]

Defining the function \( \phi(u, v) \) by

\[ \phi(u, v) = \gamma - \beta + (1 - \gamma)u + \frac{(1 - \gamma)v}{\alpha}, \tag{2.6} \]

(note that \( u = p(z) \) and \( v = zp'(z) \), we have that

(i) \( \phi(u, v) \) is continuous in \( D = C^2 \);
(ii) \( (1,0) \in D \) and \( \text{Re}\{\phi(1,0)\} = 1 - \beta > 0 \);
(iii) for all \((iu_2, v_1) \) such that \( v_1 \leq -n(1 + u_2^2)/2 \),
\[ Re\{\phi(\alpha u_2, \nu_1)\} = \gamma - \beta + \frac{(1 - \gamma)\nu_1}{\alpha} \leq \gamma - \beta - \frac{n(1 - \gamma)(1 + \nu_2^2)}{2\alpha} = -\frac{n(1 - \gamma)\nu_2^2}{2\alpha} \leq 0. \]

Therefore, the function \( \phi(u, v) \) satisfies the conditions in Lemma. This implies that \( Re\{p(z)\} > 0(z \in U) \), which is equivalent to

\[ Re\left\{\left(\frac{f(z)}{z}\right)^{\alpha}\right\} > \frac{n + 2\alpha\beta}{n + 2\alpha} \quad (z \in U). \quad (2.7) \]

This completes the assertion of Theorem 1.

Letting \( \beta = 0 \) in Theorem 1, we have

**COROLLARY 1.** If \( f(z) \in B(n, \alpha, 0) \), then

\[ Re\left\{\left(\frac{f(z)}{z}\right)^{\alpha}\right\} > \frac{n + \beta}{n + 1} \quad (z \in U). \quad (2.8) \]

**REMARK.** If we take \( \alpha = 1 \) in Corollary 1, then we have the inequality (1.3) by Cho [1].

Making \( \alpha = 1/2 \), Theorem 1 gives

**COROLLARY 2.** If \( f(z) \in B(n, 1/2, \beta) \), then

\[ Re\left\{\left(\frac{f(z)}{z}\right)^{\alpha/2}\right\} > \frac{n + \beta}{n + 1} \quad (z \in U). \quad (2.9) \]

Finally, we derive

**THEOREM 2.** If \( f(z) \in B(n, \alpha, \beta) \), then

\[ Re\left\{\left(\frac{f(z)}{z}\right)^{\alpha/2}\right\} > \frac{n + \sqrt{n^2 + 4\alpha\beta(n + \alpha)}}{2(n + \alpha)} \quad (z \in U). \quad (2.10) \]

**PROOF.** Defining the function \( p(z) \) by

\[ \left\{\frac{f(z)}{z}\right\}^{\alpha/2} = \gamma + (1 - \gamma)p(z) \quad (2.11) \]

with

\[ \gamma = \frac{n + \sqrt{n^2 + 4\alpha\beta(n + \alpha)}}{2(n + \alpha)}, \quad (2.12) \]

we easily see that \( p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots \) is regular in \( U \). Taking the differentiations of both sides in (2.11), we obtain that

\[ \frac{f'(z)f(z)^{\alpha - 1}}{z^{\alpha - 1}} = (\gamma + (1 - \gamma)p(z))^2 + \frac{2}{\beta}(1 - \gamma)(\gamma + (1 - \gamma)p(z))zp'(z) \quad (2.13) \]
that is, that
\[
\text{Re}\left\{ \frac{f(z)f(z)^\alpha - 1}{z^\alpha - 1} - \beta \right\}
\]
\[
= \text{Re}\left\{ (\gamma + (1 - \gamma)p(z))^2 + \frac{\beta}{\alpha} (1 - \gamma)(\gamma + (1 - \gamma)p(z))zp'(z) - \beta \right\}
\]
\[
> 0.
\]

Therefore, letting
\[
\phi(u, v) = (\gamma + (1 - \gamma)u)^2 + \frac{\beta}{\alpha} (1 - \gamma)(\gamma + (1 - \gamma)u)v - \beta,
\]
(note that \( p(z) = u + u_1 + iv_2 \) and \( zp'(z) = v = v_1 + iv_2 \), we observe that
(i) \( \phi(u, v) \) is continuous in \( D = C^2 \);
(ii) \( (1, 0) \in D \) and \( \text{Re}(\phi(1, 0)) = 1 - \beta > 0 \);
(iii) for all \( (iu_2, v) \in D \) such that \( v \leq n(1 + u_2^2)/2 \),
\[
\text{Re}(\phi(iu_2, v)) = \gamma^2 - (1 - \gamma)^2u_2^2 + \frac{\beta}{\alpha}(1 - \gamma)v_1 - \beta
\]
\[
\leq \gamma^2 - \beta - (1 - \gamma)^2u_2^2 - \frac{\beta}{\alpha}(1 - \gamma)(1 + u_2^2)
\]
\[
\leq 0.
\]

Thus, the function \( \phi(u, v) \) satisfies the conditions in Lemma. Applying Lemma, we conclude that
\[
\text{Re}\left\{ \frac{f(z)}{z^\alpha} \right\} > \gamma = \frac{n + \sqrt{n^2 + 4\alpha\beta(n + \alpha)}}{2(n + \alpha)} \quad (z \in U).
\]

Taking \( \alpha = 1 \) in Theorem 2, we have

COROLLARY 3. If \( f(z) \in B(n, 1, \beta) \), then
\[
\text{Re}\left\{ \frac{f(z)}{z} \right\} > \gamma = \frac{n + \sqrt{n^2 + 4n\beta + 4\beta}}{2(n + 1)} \quad (z \in U).
\]

REMARK. If we take \( \alpha = 2 \) and \( \beta = 0 \) in Theorem 2, then we have Theorem A by Cho [1].

REFERENCES
