A CLASS OF UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS

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ABSTRACT. $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is said to be in $V(\theta_n)$ if the analytic and univalent function f in the unit disc E is normalised by $f(0) = 0$, $f'(0) = 1$ and $\arg a_n = \theta_n$ for all n . If further there exists a real number β such that $\theta_n + (n-1)\beta \equiv \pi \pmod{2\pi}$ then f is said to be in $V(\theta_n, \beta)$. The union of $V(\theta_n, \beta)$ taken over all possible sequence $\{\theta_n\}$ and all possible real number β is denoted by V . $V_n(A, B)$ consists of functions $f \in V$ such that

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+Aw(z)}{1+Bw(z)},$$

$-1 \leq A < B \leq 1$, where $n \in \mathbb{N} \cup \{0\}$ and $w(z)$ is analytic, $w(0) = 0$ and $|w(z)| < 1$, $z \in E$. In this paper we find the coefficient inequalities, and prove distortion theorems.

KEY WORDS AND PHRASES. Varying arguments, Ruscheweyh derivative, Distortion theorems, Coefficient estimates.

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1. INTRODUCTION.

Let A denote the class of functions $f(z)$ analytic in the unit disc $E = \{z : |z| < 1\}$. Let S denote the subclass of A consisting functions normalised by $f(0) = 0$ and $f'(0) = 1$ which are univalent in E . The Hadamard product $(f*g)(z)$ of two functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ in A is given by,

$$(f*g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m.$$

Let $D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$, $n \in \mathbb{N} \cup \{0\}$ where $\mathbb{N} = \{1, 2, 3, \dots\}$.

Ruscheweyh [2] observed that $D^n f(z) = z(z^{n-1}f(z))^{(n)}/n!$. $D^n f(z)$ is called the n^{th} Ruscheweyh derivative of $f(z)$ by Al-Amiri [1].

DEFINITION 1. (Silverman [3]). $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is said to be in $V(\Theta_n)$ if $f \in S$ and $\arg a_n = \Theta_n$ for all n . If further there exists a real number β such that $\Theta_n + (n-1)\beta \equiv \pi \pmod{2\pi}$, then f is said to be in $V(\Theta_n, \beta)$. The union of $V(\Theta_n, \beta)$ taken over all possible sequences $\{\Theta_n\}$ and all possible real number β is denoted by V .

Now we define the class $V_n(A, B)$ consisting of functions $f \in V$ such that $\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+Aw(z)}{1+Bw(z)}$, $-1 \leq A < B \leq 1$, where $n \in \mathbb{N} \cup \{0\}$ and $w(z)$ is analytic, $w(0) = 0$ and $|w(z)| < 1$, $z \in E$. Let $K_n(A, B)$ denote the class of functions $f \in V$ such that $zf'(z) \in V_n(A, B)$.

2. COEFFICIENT INEQUALITIES.

THEOREM 1. Let $f \in V$. Then $f \in V_n(A, B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)! (m-1)!} C_m |a_m| < (B-A). \quad (2.1)$$

where $C_m = (B+1)(n+m) - (1+A)(n+1)$.

PROOF. Suppose $f \in V_n(A, B)$. Then

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \quad -1 \leq A < B \leq 1$$

$w(z)$ is analytic, $w(0) = 0$ and $|w(z)| < 1$, $z \in E$. We get

$$w(z) = \frac{D^n f(z) - D^{n+1} f(z)}{B D^n f(z) - A D^{n+1} f(z)}.$$

Since $\operatorname{Re} w(z) < |w(z)| < 1$, we obtain on simplification,

$$\operatorname{Re} \left\{ \frac{\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)! (m-1)!} [(n+1) - (n+m)] a_m z^{m-1}}{(B-A) + \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)! (m-1)!} [B(n+m) - A(n+1)] a_m z^{m-1}} \right\} < 1. \quad (2.2)$$

Since $f \in V$, f lies in $V(\Theta_m, \beta)$ for some sequence $\{\Theta_m\}$ and a real number β such that

$$\Theta_m + (m-1)\beta \equiv \pi \pmod{2\pi}. \quad \text{Set } z = re^{i\beta}.$$

Then we get,

$$\leq \left\{ \frac{\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [(n+1)-(n+m)] |a_m| r^{m-1} e^{i(\Theta_m + \overline{m-1} \beta)}}{(B-A) + \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [B(n+m)-A(n+1)] |a_m| r^{m-1} e^{i(\Theta_m + \overline{m-1} \beta)}} \right\} < 1. \quad (2.3)$$

$$\begin{aligned} & \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [(n+m)-(n+1)] |a_m| r^{m-1} \\ & < (B-A) - \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [B(n+m)-A(n+1)] |a_m| r^{m-1} \\ & \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [(B+1)(n+m)-(1-A)(n+1)] |a_m| r^{m-1} < (B-A) \end{aligned}$$

Hence,

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} C_m |a_m| r^{m-1} < (B-A). \quad (2.4)$$

Letting $r \rightarrow 1$ we get (2.1).

Conversely, suppose $f \in V$ and satisfies (2.1). In view of (2.4) which is implied by (2.1), since $r^{m-1} < 1$, we have,

$$\begin{aligned} & \left| \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [(n+1)-(n+m)] a_m z^{m-1} \right| \\ & \leq \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [(n+m)-(n+1)] |a_m| r^{m-1} \\ & < (B-A) - \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [B(n+m)-A(n+1)] |a_m| r^{m-1} \\ & \leq |(B-A) - \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} [A(n+1)-B(n+m)] a_m z^{m-1}| \end{aligned}$$

which gives (2.2) and hence follows that $f \in V_n(A, B)$.

COROLLARY 1. If $f \in V$ is in $V_n(A, B)$ then,

$$|a_m| \leq \frac{(n+1)!(m-1)!(B-A)}{(n+m-1)! C_m}$$

for $m \geq 2$. The equality holds for the function f given by,

$$f(z) = z + \frac{(n+1)!(m-1)!(B-A)}{(n+m-1)! C_m} e^{i\Theta_m} z^m, \quad z \in E.$$

THEOREM 2. Let $f \in V$. Then $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is in $K_n(A, B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} m C_m |a_m| < (B-A).$$

THEOREM 3. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in V_n(A, B)$, with $\arg a_m = \theta_m$ where $[\theta_m + (m-1)\beta] \equiv \pi \pmod{2\pi}$. Define $f_1(z) = z$ and $f_m(z) = z + \frac{(n+1)!(m-1)!(B-A)e^{i\theta_m} z^m}{(n+m-1)! C_m}$, $m = 2, 3, \dots$, $z \in E$.
 $f \in V_n(A, B)$ if and only if f can be expressed as $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ where $\mu_m \geq 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

PROOF. If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1$, $\mu_m \geq 0$, then,

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{(n+m-1)! C_m \mu_m}{(n+1)!(m-1)!} \cdot \frac{(n+1)!(m-1)!(B-A)}{(n+m-1)! C_m} \\ = \sum_{m=2}^{\infty} \mu_m (B-A) = (1-\mu_1)(B-A) \leq (B-A). \end{aligned}$$

Hence $f \in V_n(A, B)$.

Conversely, let

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in V_n(A, B),$$

define, $\mu_m = \frac{(n+m-1)! |a_m| C_m}{(n+1)!(m-1)!(B-A)}$, $m = 2, 3, \dots$ and define

$$\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m. \text{ From Theorem 1, } \sum_{m=2}^{\infty} \mu_m \leq 1 \text{ and so } \mu_1 \geq 0.$$

Since, $\mu_m f_m(z) = \mu_m z + a_m z^m$,

$$\sum_{m=1}^{\infty} \mu_m f_m(z) = z + \sum_{m=2}^{\infty} a_m z^m = f(z).$$

THEOREM 4. Define $f_1(z) = z$ and

$$f_m(z) = z + \frac{e^{i\theta_m} (n+1)!(m-1)!(B-A) z^m}{(n+m-1)! m C_m}, \quad m = 2, 3, \dots, z \in E.$$

Then $f \in K_n(A, B)$ if and only if f can be expressed as

$$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z) \text{ where } \mu_m \geq 0 \text{ and } \sum_{m=1}^{\infty} \mu_m = 1.$$

3. DISTORTION THEOREMS.

THEOREM 5. Let the function $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in the class $V_n(A, B)$. Then,

$$|z| - (B-A)|z|^2/C_2 \leq |f(z)| \leq |z| + (B-A)|z|^2/C_2 \quad (3.1)$$

$$1 - 2(B-A)|z|/C_2 \leq |f'(z)| \leq 1 + 2(B-A)|z|/C_2. \quad (3.2)$$

$$\text{PROOF. } |f(z)| = |z + \sum_{m=2}^{\infty} a_m z^m| \leq |z| + |z|^2 \sum_{m=2}^{\infty} |a_m|$$

and $|f(z)| \geq |z| - |z|^2 \sum_{m=2}^{\infty} |a_m|$. Since $\frac{(n+m-1)! C_m}{(n+1)!(m-1)!}$ is an increasing function of $m \geq 2$ and $f(z) \in V_n(A, B)$, by Theorem 1, we have

$$\frac{(n+1)!}{(n+1)! 1!} C_2 \sum_{m=2}^{\infty} |a_m| \leq \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} C_m |a_m| \leq (B-A)$$

that is,

$$\sum_{m=2}^{\infty} |a_m| \leq \frac{B-A}{C_2} \quad (3.3)$$

From (3.3) we get (3.1)

$$|f'(z)| = \left| 1 + \sum_{m=2}^{\infty} m a_m z^{m-1} \right| \leq 1 + |z| \sum_{m=2}^{\infty} m |a_m|$$

and

$$|f'(z)| \geq 1 - |z| \sum_{m=2}^{\infty} m |a_m|.$$

Since $\frac{(n+m-1)! C_m}{(n+1)! m!}$ is an increasing function of $m \geq 2$ and

$$\frac{(n+m-1)! m C_m}{(n+1)! (m+1)!} < \frac{(n+m-1)! m C_m}{(n+1)! m!} \text{ by Theorem 1, we have,}$$

$$\frac{(n+1)! C_2}{(n+1)! 2} \sum_{m=2}^{\infty} m |a_m| \leq \sum_{m=2}^{\infty} \frac{(n+m-1)! C_m |a_m|}{(n+1)! (m-1)!} \leq (B-A)$$

that is,

$$\sum_{m=2}^{\infty} m |a_m| \leq \frac{2(B-A)}{C_2}. \quad (3.4)$$

From (3.4) we get (3.2). Further for the function $f(z) = z + \frac{(B-A)}{C_2} z^2$, we can see that the results of the Theorem are sharp.

COROLLARY 2. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in the class $V_n(A, B)$. Then $f(z)$ is included in a disc with its center at the origin and radius r given by $r = (C_2 + B - A)/C_2$ and $f'(z)$ is included in a disc with its center at the origin and radius r_1 given by $r_1 = [C_2 + 2(B - A)]/C_2$.

THEOREM 6. Let the function $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in the class $K_n(A, B)$, then,

$$|z| - (B-A)|z|^2/2 C_2 \leq |f(z)| \leq |z| + (B-A)|z|^2/2 C_2$$

and

$$1 - (B-A)|z|/C_2 \leq |f'(z)| \leq 1 + (B-A)|z|/C_2$$

for $z \in E$. The results are sharp for the function $f(z) = z + (B-A)z^2/2 C_2$.

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