SPACES OF COMPACT OPERATORS WHICH ARE M -IDEALS IN $L(X, Y)$

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ABSTRACT. Suppose X and Y are reflexive Banach spaces. If $K(X, Y)$, the space of all compact linear operators from X to Y is an M -ideal in $L(X, Y)$, the space of all bounded linear operators from X to Y , then the second dual space $K(X, Y)^{**}$ of $K(X, Y)$ is isometrically isomorphic to $L(X, Y)$.

KEY WORDS AND PHRASES. Compact operators, M -ideal, dual space, projective tensor product.

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1. INTRODUCTION

It is well known that if X and Y are reflexive Banach spaces one of which satisfies the approximation property then the second dual space $K(X, Y)^{**}$ of $K(X, Y)$, the space of compact linear operators from X to Y , is isometrically isomorphic to $L(X, Y)$, the space of bounded linear operators from X to Y (Diestel [1, p.17]). Harmand and Lima [2] proved that if X is a reflexive Banach space and $K(X)$ is an M -ideal in $L(X)$ then $K(X)^{**}$ is isometrically isomorphic to $L(X)$.

The purpose of this paper is to generalize the result of Harmand and Lima to the case of $K(X, Y)$ and $L(X, Y)$ by modifying their proof. In Theorem 3.3 we will prove that if X and Y are reflexive Banach spaces and $K(X, Y)$ is an M -ideal in $L(X, Y)$ then $K(X, Y)^{**}$ is isometrically isomorphic to $L(X, Y)$.

2. NOTATIONS AND PRELIMINARIES.

Let X and Y be Banach spaces. $X \simeq Y$ means that X and Y are isometrically isomorphic. $L(X, Y)$ (resp. $K(X, Y)$) will denote the space of all bounded linear operators (resp. compact linear operators) from X to Y . If $X = Y$, then we simply write $L(X)$ (resp. $K(X)$). X^* will denote the dual space of X and we will write $\langle x, x^* \rangle$ for the action of $x^* \in X^*$ on $x \in X$ instead of $x^*(x)$. B_X will denote the closed unit ball of X .

A closed subspace J of a Banach space X is called an L -summand if there exists a projection P on X such that $PX = J$ and $\|x\| = \|Px\| + \|x - Px\|$ for every x in X . In this case we write $X = J \oplus_1 J'$ where $J' = (I - P)X$. A closed subspace J of a Banach space X is called an M -ideal in X if J° , the annihilator of J in X^* , is an L -summand in X^* .

Let $X \hat{\otimes} Y$ be the projective tensor product of Banach spaces X and Y . If $u \in X \hat{\otimes} Y$, then there exist sequences (x_i) in X and (y_i) in Y such that $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, with $\sum_{i=1}^{\infty} \|x_i\| \|y_i\| < \infty$. Moreover, we have $\|u\| = \inf \sum_{i=1}^{\infty} \|x_i\| \|y_i\| < \infty$, the infimum being taken over all representations $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, $x_i \in X$, $y_i \in Y$ and $\sum_{i=1}^{\infty} \|x_i\| \|y_i\| < \infty$. (Diestel and Uhl [3, p. 227]).

Let Z be another Banach space and $T \in L(X, Z)$. We define $Tu = \sum_{i=1}^{\infty} x_i \otimes Ty_i$ for $u = \sum_{i=1}^{\infty} x_i \otimes y_i \in X \hat{\otimes} Y$. Then $Tu \in X \hat{\otimes} Z$ and $\|Tu\| \leq \|T\| \|u\|$. If $u = \sum_{i=1}^{\infty} x_i^* \otimes x_i \in X^* \hat{\otimes} X$ with $\sum_{i=1}^{\infty} \|x_i^*\| \|x_i\| < \infty$, the map $u \mapsto \text{tr}(u) = \sum_{i=1}^{\infty} \langle x_i^*, x_i^* \rangle$ defines a bounded linear functional on $X^* \hat{\otimes} X$ with norm no larger than 1.

THEOREM 2.1 (Diestel and Uhl [3], Shatten [4]). Let X and Y be Banach spaces. The Banach space $L(X, Y^*)$ is isometrically isomorphic to $(Y \hat{\otimes} X)^*$ and under this identification $T \in L(X, Y^*)$ act on $u \in Y \hat{\otimes} X$ by $\langle u, T \rangle = \text{tr}(Tu)$.

THEOREM 2.2 (Feder and Sapher [5]). Let X and Y be Banach spaces. If either X^{**} or Y^* has the Radon-Nikodym Property, then map $V: Y^* \hat{\otimes} X^{**} \rightarrow K(X, Y)^*$ defined by $\langle T, V(u) \rangle = \text{tr}(T^{**}u)$ for $T \in K(X, Y)$ and $u \in Y^* \hat{\otimes} X^{**}$ is a quotient map.

3. SPACES OF COMPACT OPERATORS

Harmand and Lima [2] proved that if $K(X)$ is an M -ideal in $L(X)$ then there exists a net (T_α) in $B_{K(X)}$ such that

- (i) $T_\alpha x \rightarrow x$ for all $x \in X$
- (ii) $T_\alpha^* x^* \rightarrow x^*$ for all $x^* \in X^*$
- (iii) $\|T_\alpha - I\| \rightarrow 1$.

In the case of $K(X, Y)$ and $L(X, Y)$, we have the following analogue which also plays a key role in the proof of our main result (Theorem 3.3).

THEOREM 3.1. If X and Y are Banach spaces and $K(X, Y)$ is an M -ideal in $L(X, Y)$, then for each T in $B_{L(X, Y)}$ there is a net (T_α) in $B_{K(X, Y)}$ such that

- (i) $T_\alpha x \rightarrow Tx$ for all $x \in X$
- (ii) $T_\alpha^* y^* \rightarrow T^* y^*$ for all $y^* \in Y^*$.

PROOF. Suppose $K(X, Y)$ is an M -ideal in $L(X, Y)$. Then we can write $L(X, Y)^* = K(X, Y)^\circ \oplus_1 J$ for some subspace J of $L(X, Y)^*$.

The map $\phi \rightarrow \phi + K(X, Y)^\circ$ defines an isometry from J onto $L(X, Y)^*/K(X, Y)^\circ$ and the map $\phi + K(X, Y)^\circ \rightarrow \phi|_{K(X, Y)}$ defines an isometry from $L(X, Y)^*/K(X, Y)^\circ$ onto $K(X, Y)^*$ (Rudin [6, p.91]). Hence the map $\phi \rightarrow \phi|_{K(X, Y)}$ gives an isometry from J onto $K(X, Y)^*$.

Let Q be the projection on $L(X, Y)^*$ with the range J . Then $\phi \in L(X, Y)^*$ is in the range of Q if and only if the restriction of ϕ to $K(X, Y)$ has the same norm as ϕ . If $T \in L(X, Y) \subseteq L(X, Y)^{**}$ with $\|T\| \leq 1$, then for $\phi \in K(X, Y)^\circ$ we have $(Q^*T)\phi = TQ(\phi) = 0$ thus $Q^*T \in K(X, Y)^\circ = J^\perp = K(X, Y)^{**}$. Since $Q^*T \in K(X, Y)^{**}$ and $\|Q^*T\| \leq 1$, by the Goldstein's theorem there is a net (T_α) in $B_{K(X, Y)}$ such that

$$T_\alpha \rightarrow Q^*T \text{ in the weak}^*\text{-topology on } J^\perp = K(X, Y)^{**}.$$

We claim that $T_\alpha x \rightarrow Tx$ for all $x \in X$ and $T_\alpha^* y^* \rightarrow T^* y^*$ for all $y^* \in Y^*$. For $x^{**} \in X^{**}$ and $y^* \in Y^*$, define $\phi_{x^{**} \otimes y^*} \in L(X, Y)^*$ by

$$\langle A, \phi_{x^{**} \otimes y^*} \rangle = \langle A^* y^*, x^{**} \rangle.$$

Then we can easily see that $\phi_{x^{**} \otimes y^*} \in J = K(X, Y)^*$ and hence

$$\langle T_\alpha^* y^*, x^{**} \rangle \rightarrow \langle T^* y^*, x^{**} \rangle.$$

By the weak*-compactness of $B_{X^{**}}$ we get that

$$T_\alpha^* \rightarrow T^* y^* \text{ for all } y^* \in Y^*.$$

Similarly, for $y^* \in Y^*$ and $x \in X$ the functional $\psi_{y^*} \otimes x$ on $L(X, Y)$ defined by $\langle A, \psi_{y^*} \otimes x \rangle = \langle Ax, y^* \rangle$ for $A \in L(X, Y)$ is in the range of Q and hence $T_\alpha x \rightarrow Tx$ for all $x \in X$.

The following proposition is essentially due to Harmand and Lima [2] who treated a special case $X = Y$.

PROPOSITION 3.2. Let X and Y be Banach spaces and V the map defined in Theorem 2.2. If $K(X, Y)$ is an M -ideal in $L(X, Y)$, then $T^{**} \in (\ker V)^\circ$ for every $T \in L(X, Y)$.

PROOF. Recall that by Theorem 2.1 we have $(Y^* \hat{\otimes} X^{**})^* \simeq L(X^{**}, Y^{**})$ and under this identification $S \in L(X^{**}, Y^{**})$ acts on $u \in Y^* \hat{\otimes} X^{**}$ by $\langle u, S \rangle = \text{tr}(Su)$.

Let $T \in L(X, Y)$, $\|T\| \leq 1$. By Theorem 3.1 there is a net (T_α) in $B_{K(X, Y)}$ such that $T_\alpha^* y^* \rightarrow T^* y^*$ for all $y^* \in Y^*$. Let $u = \sum_{i=1}^{\infty} y_i^* \otimes x_i^{**} \in \ker V$ with $\sum_{i=1}^{\infty} \|y_i^*\| \|x_i^{**}\| < \infty$. We may assume that $\|x_i^{**}\| \leq 1$ for all i and

$\|y_i^*\| \rightarrow 0$. Then we get

$$\begin{aligned} 0 &= \langle T_\alpha, V(u) \rangle \\ &= \text{tr}(T_\alpha^{**}u) \\ &= \sum_{i=1}^{\infty} \langle y_i^*, T_\alpha^{**}x_i^{**} \rangle \\ &= \sum_{i=1}^{\infty} \langle T_\alpha^* y_i^*, x_i^{**} \rangle \\ &= \sum_{i=1}^{\infty} \langle T^* y_i^*, x_i^{**} \rangle \\ &= \text{tr}(T^{**}u) \\ &= \langle u, T^{**} \rangle. \end{aligned}$$

Thus $T^{**} \in (\ker V)^\circ$.

THEOREM 3.3. If X and Y are reflexive Banach spaces and $K(X, Y)$ is an M -ideal in $L(X, Y)$ then $K(X, Y)^{**}$ is isometrically isomorphic to $L(X, Y)$.

PROOF. Since X and Y are reflexive, X and Y^* have the Radon-Nikodym property and hence by Theorem 2.2 the map $V: Y^* \hat{\otimes} X^{**} \rightarrow K(X, Y)^*$ defined by

$$\langle T, V(u) \rangle = \text{tr}(T^{**}u) \quad \text{for } u \in Y^* \hat{\otimes} X^{**}, T \in K(X, Y)$$

is a quotient map. Thus $V^*: K(X, Y)^{**} \rightarrow (Y^* \hat{\otimes} X^{**})^*$ is an isometry with the range $(\ker V)^\circ$ and hence we have

$$\begin{aligned} K(X, Y)^{**} &\simeq (\ker V)^* \\ &\subseteq (Y^* \hat{\otimes} X^{**})^* \\ &\simeq L(X^{**}, Y^{**}) \\ &= L(X, Y). \end{aligned}$$

Since X and Y are reflexive, $T = T^{**}$ for all $T \in L(X, Y)$ and by Proposition 3.2 $(Y^* \hat{\otimes} X^{**})^* \subseteq (\ker V)^\circ$. Thus $K(X, Y)^{**} \simeq L(X, Y)$.

Recall that for $1 \leq p \leq \infty$ the l_p -sum $(\Sigma X_n)_p$ of a sequence of (X_n) of Banach spaces is the Banach space of all sequences (x_n) with $x_n \in X_n$ and with the norm $\|(x_n)\| = (\Sigma \|x_n\|^p)^{1/p} < \infty$.

COROLLARY 3.4. Suppose X and Y are closed subspaces of $(\Sigma X_n)_p$ and $(\Sigma Y_n)_q$ ($1 < p \leq q \leq \infty$, $\dim X_n < \infty$, $\dim Y_n < \infty$), respectively. If $K(X, Y)$ is dense in $L(X, Y)$ in the strong operator topology, then $K(X, Y)^{**} \simeq L(X, Y)$.

PROOF. X and Y are reflexive and $K(X, Y)$ is an M -ideal in $L(X, Y)$ (Cho [7]).

REMARK. If X and Y are as in Corollary 3.4 and either X or Y satisfies the compact approximation property, then $K(X, Y)$ is dense in $L(X, Y)$ in the strong operator topology [7] and hence $K(X, Y)^{**} \simeq L(X, Y)$.

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REFERENCES

1. DIESTEL, J. Geometry of Banach Spaces-Selected Topics, Lecture Note in Math., Vol. 485, Springer-Verlag, Berlin and New York, 1975.
2. HARMAND, P. and LIMA, A., Banach Spaces which are M -ideals in their Biduals, Trans. Amer. Math. Soc. **238** (1983), 253-264.
3. DIESTEL, J. and UHL, J., Vector Measures, American Mathematical Survey No. 15(1977).
4. SHATTEN, R., A Theory of Cross-space, Ann. of Math. Studies **26**, 1950.
5. FEDER, M. and SAPHAR, P., Spaces of Compact Operators and their Dual Spaces, Israel J. Math. **21** (1975), 38-49.
6. RUDEN, W., Functional Analysis, McGraw-Hill Book Company, 1973.
7. CHO, C. M., A Note on M -ideals of Compact Operators, Canadian Math Bull. **32**(1989), 434-440.