LINEAR FUNCTIONALS ON ORLICZ SEQUENCE SPACES WITHOUT LOCAL CONVEXITY

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ABSTRACT. The general form of continuous linear functionals on an Orlicz sequence space $1^*$ (non-separable and non-locally convex in general) is obtained. It is proved that the space $h^*$ is an $M$-ideal in $1^*$.

KEY WORDS AND PHRASES. Orlicz sequence spaces, Köthe dual, Riesz spaces, Mackey topologies, modular spaces, and $M$-ideals.

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INTRODUCTION. The general form of continuous linear functionals on an Orlicz space $L^*$, defined by a convex Orlicz function $\Phi$ has been found by Ando [2] (for $\Phi$ being an $N$-function and for a finite measure space) and by Rao [21], Fernandez [7] (for $\Phi$ being a Young function and for a general measure space).

In this paper we describe the dual space $(1^*)'$ of an Orlicz sequence space $1^*$ defined by an arbitrary Orlicz function $\Phi$ (not necessarily convex) such that $\Phi(u)/u \to \infty$ as $u \to \infty$. For this purpose we shall first use the description of the Mackey topology $\tau_{1^*}$, obtained by Kalton [8], when $\Phi$ satisfies the $A_2$-condition at 0, and by Drewnowski and Nawrocki [5], in general. The Mackey topology $\tau_{1^*}$ is normable and we consider two natural norms on $1^*$ which generate $\tau_{1^*}$. Thus we can define two corresponding norms in $(1^*)'$. Moreover, we consider $1^*$ from the point of view of the theory of modular spaces (see [15], [16], [17]). We investigate the conjugate modular (in the sense of Nakano [17]) on $(1^*)'$ and consider two other norms on $(1^*)'$ defined in a natural way by the conjugate modular. It is well-known that $(1^*)' = (1^*) + (1^*)'$, where $(1^*)$ and $(1^*)'$ denote the sets of all order continuous and singular linear functionals on $1^*$ respectively. We first show that the Köthe dual $(1^*)'$ of $1^*$ coincides with the Orlicz sequence space $1^*$, where $\Phi^*$ denotes the complementary function of $\Phi$ in the sense of Young. Thus we obtain the corresponding characterization of $(1^*)_+$. Next, we prove that the conjugate modular and all four norms defined on $(1^*)_+$ coincide on $(1^*)_+$. Following the idea of [2] we construct a Riesz isometric isomorphism of $(1^*)_+$ onto some Riesz subspace $B_4(N)$ (dependent on $\Phi$) of the Banach lattice $ba(N)$ of all real-valued bounded finitely additive set functions on $N$. We prove that there exists an isometric isomorphism of the Banach space $((1^*)'_\Phi, \|\cdot\|_\Phi)$ (for the definition of the norm $\|\cdot\|_\Phi$ see section 2) onto the Banach space $1^* \times B_4(N)$ given by the mapping $f \mapsto (y, \nu)$ such that $f(x) = \sum_{i=1}^\infty x(i)y(i) + f(x)d\nu$ for all $x \in 1^*$ and $\|f\|_\Phi = \|y\|_\Phi + |\nu| (N)$. From this it follows that $h^*$ (the ideal of elements of absolutely continuous $F$-norm on $1^*$) is an $M$-ideal of $1^*$ (see [3, definition 2.1]). As an application, we obtain that every continuous linear function on $h^*$ has the unique norm preserving extension to $1^*$.

1. Preliminaries. For terminology concerning locally solid Riesz spaces we refer to [1] and [14]. For a Riesz space $(E, \geq)$ let $E^* = \{u \in E : u \geq 0\}$ (the positive cone of $E$). By $N$ we will denote the set of all natural numbers. Denote by $\omega$ the space of all real-valued sequences. For the sequence $x$, $x(i)$ means the
The $i$-th coordinate of $x$, and we shall denote by $x^{(i)}$ the $n$-th section of $x$ (that is $x^{(i)}(i) = x(i)$ for $i \leq n$, $x^{(i)}(i) = 0$ for $i > n$). For a subset $A$ of $N$ we will denote by $x_A$ the sequence such that $x_A(i) = x(i)$ for $i \in A$ and $x_A(i) = 0$ for $i \not\in A$. If $f$ is a linear functional on a subspace $X$ of $\omega$, we will denote by $f_A$ the functional defined as: $f_A(x) = f(x_A)$ for $x \in X$. It is known that $\omega$ is a super Dedekind complete Riesz space under the ordering $x \leq y$ whenever $x(i) \leq y(i)$ for $i \in N$.

Now we recall some terminology concerning Orlicz sequence spaces (see [11], [12], [22], and [25]).

By an Orlicz function $\phi$ we mean a function $\phi: [0, \infty) \to [0, \infty)$ which is non-decreasing, continuous for $u > 0$ and $\phi(u) = 0$ iff $u = 0$. Throughout this paper we shall assume that $\phi$ satisfies the following condition: $\phi(u)/u \to \infty$ as $u \to \infty$. Every Orlicz function $\phi$ determines the functional $\rho_\phi: \omega_\phi \to [0, \infty]$ defined by the formula:

$$
\rho_\phi(x) = \sum_{i=1}^{\infty} \phi(|x(i)|).
$$

Then $1^\phi = \{x \in \omega : \rho_\phi(\lambda x) < \infty$ for some $\lambda > 0\}$ is called an Orlicz sequence space defined by $\phi$. The space $1^\phi$ is an ideal of $\omega$ and the functional $\rho_\phi$ restricted to $1^\phi$ is an orthogonal additive modular, i.e., $\rho_\phi$ satisfies the following conditions:

1. $\rho_\phi(x) = 0$ iff $x = 0$.
2. $\rho_\phi(x_1) = \rho_\phi(x_2)$ if $|x_1| \leq |x_2|$.
3. $\rho_\phi(\lambda x) \to 0$ if $\lambda \to 0$.
4. $\rho_\phi(x_1 + x_2) = \rho_\phi(x_1) + \rho_\phi(x_2)$ if $|x_1| \wedge |x_2| = 0$.

These conditions imply that $\rho_\phi(x_1 \vee x_2) \leq \rho_\phi(x_1) + \rho_\phi(x_2)$ for $x_1, x_2 \geq 0$. Moreover, $\rho_\phi$ satisfies the following axiom of completeness (see [15]):

(C) If $x_n = 0$ for $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} \rho_\phi(x_n) < \infty$, then there exists $y \in 1^\phi$ such that $y = \sup x_n$ and $\rho_\phi(y) = \sum_{n=1}^{\infty} \rho_\phi(x_n)$.

If $\phi$ is a convex Orlicz function, then the modular $\rho_\phi$ is convex, i.e.,

$$
\rho_\phi(ax_1 + bx_2) \leq a \rho_\phi(x_1) + b \rho_\phi(x_2) \text{ for } a, b > 0 \text{ with } a + b = 1.
$$

In $1^\phi$ the complete Riesz $F$-norm $\| \cdot \|_\phi$ can be defined by

$$
\|x\|_\phi = \inf\{\lambda > 0 : \rho_\phi(\lambda x) \leq \lambda\}.
$$

We shall denote by $\tau_\phi$ the topology of the $F$-norm $| \cdot |_\phi$. Let $h^\phi = \{x \in 1^\phi : \rho_\phi(\lambda x) < \infty$ for all $\lambda > 0\}$. Then $h^\phi$ is the ideal of elements of absolutely continuous $F$-norm $| \cdot |_\phi$ on $1^\phi$.

We say that $\phi$ satisfies the $\Delta_2$-condition at 0, whenever $\limsup_{s \to 0} \phi(2u)/\phi(u) < \infty$. It is known that $1^\phi = h^\phi$ (i.e. $1^\phi$ is separable) iff $\phi$ satisfies the $\Delta_2$-condition at 0.

We say that two Orlicz functions $\phi$ and $\psi$ are equivalent at 0, in symbols $\phi \sim \psi$, if there exist positive numbers $a, b, c, d$ and $\omega_0 > 0$ such that $a\phi(bu) \leq \psi(u) \leq c\phi(du)$ for $0 \leq u \leq \omega_0$. It is well-known that if $\phi \sim \psi$ then $1^\phi = 1^\psi$ and $\tau_\phi = \tau_\psi$. Moreover, the space $(1^\phi, \tau_\phi)$ is locally convex iff there exists a convex Orlicz function $\psi$ such that $\phi \sim \psi$ (see [25], Theorem 3.1.5]. Separable Orlicz sequence spaces without local convexity have been investigated in detail by Kalton [8]. For examples of non-separable and non-locally convex Orlicz sequence spaces see [5].

We denote by $\rho_\phi$ the Minkowski functional of the absolutely convex absorbing subset $k^\phi = \{x \in \omega : \rho_\phi(x) < \infty\}$ of $1^\phi$. Thus

$$
\rho_\phi(x) = \inf\{\lambda > 0 : \rho_\phi(\lambda x) < \infty\}
$$

for all $x \in 1^\phi$, $\rho_\phi(x) \leq |x|_\phi$, for $x \in 1^\phi$, and $h^\phi = \ker \rho_\phi$. 


2. Norms on the dual space \((l^r)^*\) of \(l^r\). In this section we define in two different ways some natural norms on \((l^r)^*\). For this purpose we shall first use the description of the Mackey topology of \((l^r, \tau_r)\) given in [5], and next, we apply the Nakano’s theory of conjugate modulars [17].

Let us put
\[
\phi^*(v) = \sup \{uv - \phi(u) : u \geq 0\} \text{ for } v \geq 0.
\]
Then \(\phi^*\) will be called the function complementary to \(\phi\) in the sense of Young. It is seen that \(\phi^*\) is a convex function, taking only finite values, and \(\phi^*(0) = 0\). This means that \(\phi^*\) is a Young function (see [12], [13], [26]). The additional properties of \(\phi^*\) are included in the following

**Lemma 2.1.** (a) If \(\liminf \phi(u)/u \to 0\), then \(\phi^*\) vanishes only at 0 and \(\lim \phi^*(v)/v = 0\), \(\lim \phi^*(v)/v = \infty\) (i.e. \(\phi^*\) is an N-function in the sense of [11]).

(b) If \(\liminf \phi(u)/u > 0\), then \(\phi^*\) vanishes near zero and \(\lim \phi^*(v)/v = \infty\) (i.e. \(l^r = l^s\)).

**Proof.** (a) We can easily verify that \(\phi^*(v) > 0\) for \(v > 0\). In the same way as in [4, §2] we can show that \(\lim \phi^*(v)/v = 0\) and \(\lim \phi^*(v)/v = \infty\).

(b) We shall show that there exists \(v_0 > 0\) such that \(\phi^*(v) = 0\) for \(0 \leq v \leq v_0\), and \(\phi^*(v) > 0\) for \(v > v_0\). Indeed, since \(\liminf \phi(u)/u > 0\) there exist numbers \(v' > 0\) and \(u' > 0\) such that \(uv' \leq \phi(u)\) for \(0 \leq u \leq u'\), and since \(\lim \phi(u)/u = \infty\) (by our assumption) there exists a number \(u'' > 0\) with \(u'' > u'u\). Taking \(v'' > 0\) such that \(1/v'' = \liminf \phi(u)/u\), we have \(uv'' \leq \phi(u)\) for \(u'' \leq u \leq u''\). Then for \(v = \min(1,v',v'')\) we get \(uv = uv' \leq \phi(u)\) for \(u > u''\), \(uv_1 \leq uv'' \leq \phi(u)\) for \(u'' \leq u \leq u''\), and \(uv_1 \leq u \leq \phi(u)\) for \(u > u''\). Hence \(uv_1 = \phi(u)\) for \(u = 0\), so that \(\phi^*(v_1) = 0\). On the other hand, there exists a number \(v_2 > 0\) such that \(\phi^*(v_2) > 0\). Since \(\phi^*\) is convex, there exists a number \(v_0 > 0\) such that \(\phi^*(v) = 0\) for \(0 \leq v \leq v_0\), and \(\phi^*(v) > 0\) for \(v > v_1\). Moreover, as in [4, §2] we can show that \(\lim \phi^*(v)/v = \infty\).

For an Orlicz function \(\phi\) we shall denote by \(\hat{\phi}\) the convex minorant of \(\phi\) in a neighborhood of 0, i.e., \(\hat{\phi}\) is the largest Orlicz function such that \(\hat{\phi}(u) \leq \phi(u)\) for \(u \geq 0\), and \(\hat{\phi}\) is convex on the interval [0,1] (see [8, p. 255]).

Moreover, let us put
\[
\bar{\phi}(u) = (\phi^*)^*(u) \text{ for } u \geq 0.
\]
It is seen that \(\bar{\phi}\) is a convex Orlicz function such that \(\lim_{u \to 0} \bar{\phi}(u)/u = \infty\). The relation between \(\hat{\phi}\) and \(\bar{\phi}\) is described by

**Lemma 2.2.** We have \(\hat{\phi} - \overline{\phi} = \phi(u)\) for \(u \geq 0\).

**Proof.** First, we shall show that \(\bar{\phi}(u) \leq \phi(u)\) for \(u \geq 0\). Indeed, since \(\lim \phi^*(v)/v = \infty\), for every \(u > 0\) there exists \(v_u > 0\) such that \(\bar{\phi}(u) + \phi(v_u) = uv_u\). But \(uv_u \leq \phi(u) + \phi(v_u)\); hence \(\bar{\phi}(u) \leq \phi(u)\) for \(u \geq 0\). In [18, Lemma 2.1] it is proved that \(\hat{\phi} - \bar{\phi}\) whenever \(\liminf \phi(u)/u = 0\). Now assume that \(\liminf \phi(u)/u > 0\). We can check that \(\hat{\phi} - \chi_1\), where \(\chi_1(u) = u\) for \(u \geq 0\) (see [18]). It suffices to show that \(\hat{\phi} - \chi_1\). In view of Lemma 2.1 there exists a number \(v_0 > 0\) such that \(\phi^*(v) = 0\) for \(0 \leq v \leq v_0\), and \(\phi^*(v) > 0\) for \(v > v_0\). Moreover, since \(\lim \phi^*(v)/v = \infty\), for every \(u > 0\) there exists \(v_u > v_0\) such that \(uv - \phi^*(v) < 0\) for \(v > v_u\). Hence, for every \(u > 0\), \(\bar{\phi}(u) = \max(uv_0, \sup\{uv - \phi^*(v) : v_0 \leq v \leq v_u\})\). But \(\sup\{uv - \phi^*(v) : v_0 \leq v \leq v_u\} = uv' - \phi^*(v')\) for some \(v'\) with \(v_0 \leq v' \leq v_u\). Assuming that \(v_0 < v'\), we obtain that \(\phi(u) = uv_0\) for \(0 \leq u \leq u_0 = \phi^*(v')/(v' - v_0)\), and thus \(\hat{\phi} - \chi_1\).
For a topological vector space \((E, \tau)\) we shall denote by \((E, \xi)^*\) its topological dual. We shall denote by \((1^*)^*\) the dual space of \((1^*, \tau_q)\).

Let us recall that the Mackey topology of \((E, \xi)\) is the finest locally convex topology \(\tau\) which produces the same continuous linear functionals as the original topology \(\xi\). If \((E, \xi)\) is an F-space then \(\tau\) is the finest locally convex topology on \(E\) which is weaker than \(\xi\) (see [24]).

Kalton [8] has showed that the Mackey topology \(\tau_q\) of a separable Orlicz sequence space \(1^*\) coincides with the topology \(\tau_{q_1,},\) induced from \(1^*\). For an arbitrary \(1^*\), the Mackey topology \(\tau_q\) has been described by Drewnowski and Nawrocki [5].

Denote by \(\tau_q\) the Mackey topology of \((1^*, \tau_q)\), by \(\tau_{q_1,}\) the Mackey topology of \((h^*, \tau_{q_1,})\), and by \(\tau_{q_1,}\) the topology defined by the Riesz seminorm \(p_q\).

Combining [5, Theorems 5.1 and 5.3] with Lemma 2.2 we get the following important descriptions of \(\tau_{q_1,}\) and \(\tau_q\).

**THEOREM 2.3.** The following equalities hold:
\[\tau_q = \tau_{q_1,} \quad \text{and} \quad \tau_q = (\tau_{q_1,})^\vee \tau_{q_1,}.\]

It is well-known (see [11], [12]) that the F-norm topology \(\tau_q\) on \(\hat{1}^*\) can be generated by two Riesz norms:
\[\|x\|_{q,} = \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} (\rho_q(\lambda x) + 1) \right\} \]
\[= \sup \left\{ \left| \sum_{i=1}^n x(i)z(i) \right| : x \in 1^*, \rho_q(x) \leq 1 \right\},\]
and
\[\|x\|_{q_1,} = \inf \{ \lambda > 0 : \rho_q(x/\lambda) \leq 1 \} \cdot\]
Moreover, \(\|x\|_{q_1,} \leq \|x\|_{q,} \leq 2 \|x\|_{q_1,}\) for all \(x \in \hat{1}^*\) and \(\|x\|_{q_1,} \leq 1\) iff \(\rho_q(x) \leq 1\).

Therefore, in view of Theorem 2.3 the Mackey topology \(\tau_q\) can be generated by two Riesz norms:
\[P_q \vee \|x\|_{q_1,} \quad \text{and} \quad P_q \vee \|x\|_{q_1,}\]
which will be of importance in our discussion. Thus two corresponding Riesz norms on \((1^*)^*\) can be given by
\[\|f\|_{q,}^* = \sup \{|f(x)| : x \in 1^*, \rho_q(x) \leq 1 \} \quad \text{and} \quad \|x\|_{q,} \leq 1\]
\[\|f\|_{q_1,}^* = \sup \{|f(x)| : x \in 1^*, \rho_q(x) \leq 1 \} \quad \text{and} \quad \|x\|_{q_1,} \leq 1\].

Thus \((1^*)^*\) is a Banach lattice under each of the norms \(\|x\|_{q,}^*\) and \(\|x\|_{q_1,}^*\). Moreover, since \(\rho_q(x) \leq 1\) implies \(p_q(x) \leq 1\) and \(\rho_q(x) \leq 1\), we can put (see [19]):
\[\|f\|_{q,}^* = \sup \{|f(x)| : x \in 1^*, \rho_q(x) \leq 1 \}.\]

We shall denote by \((1^*)^\diamond\) the collection of all order bounded linear functionals on \(1^*\). It is well-known that \((1^*)^\diamond = (1^*)^*\) (see [1, Theorem 16.9]). An order bounded linear functional \(f\) on \(1^*\) is said to be order continuous (resp. singular) if \(x_n \to 0\) in \(1^*\) implies \(f(x_n) \to 0\) for a net \((x_n)\) in \(1^*\) (resp. \(f(x) = 0\) for all \(x \in h^*\)) (see [9, Ch. X]). The set of all order continuous (resp. singular) functionals on \(1^*\) will be denoted by \((1^*)^-\) (resp. \((1^*)_s\)).

The next theorem gives a characterization of the space \((1^*)^*\).

**THEOREM 2.4.** (a) For a linear functional \(f\) on \(1^*\) the following statements are equivalent:
(1) \( f \) is order bounded.

(2) \( f \) is \( \tau_\phi \)-continuous.

(3) There exist unique \( f_+ \in (1^*_+)' \) and \( f_- \in (1^*_-)' \) such that
\[
  f(x) = f_+(x) + f_-(x) \quad \text{for } x \in 1^*.
\]

(b) \((1^*_+)' = (1^*_+)^d\) (= the disjoint complement of \((1^*_+)'\) in \((1^*)'\)), and moreover, \((1^*_+)\) and \((1^*_-)\) are Banach lattices under each of the norms \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_{\infty} \).

**Proof.** (a) Since \((1^*_+, \rho_\phi \vee \| \cdot \|_{\infty}^d) = (1^*)^d \equiv (1^*)_d\), by [9, Ch. VI, §1, Theorem 5], we obtain that \((1^*_+)\) separates the points of \(1^*\), and to get our result it suffices to use Theorem 6 of [9, Ch. X, §3].

(b) Since \((1^*_+)\) is a band of \((1^*)^d\) (see [1, Theorem 3.7]) \((1^*_+)\) is a \( \| \cdot \|_{\infty} \)-closed (resp. \( \| \cdot \|_{\infty} \)-closed) subspace of \((1^*)^d\) (see [1, Theorem 5.6]). Thus \((1^*_+)\) is a Banach lattice, because \((1^*)^d\) is a Banach lattice. Moreover, since \((1^*_+) = (1^*_+)\), \((1^*_+)\) is a band of \((1^*)^d\) (see [1, p. 27]), and by the above argument \((1^*_+)\) is a Banach lattice.

In view of [17] the conjugate \(\widetilde{\rho}_\phi\) of the modular \(\rho_\phi\) can be defined on the algebraic dual \(1^*\) of \(1^*\) as follows:
\[
  \widetilde{\rho}_\phi(f) = \sup \{ |f(x)| - \rho_\phi(x) : x \in 1^* \}.
\]
Note that if \(f \geq 0\), then
\[
  \widetilde{\rho}_\phi(f) = \sup \{ f(x) - \rho_\phi(x) : 0 \leq x \in \omega, \rho_\phi(x) < \infty \}.
\]
Indeed, since \( |f(x)| \leq f(|x|) \) (see [1, p. 21]) and \( \rho_\phi(x) = \rho_\phi(|x|) \) we have
\[
  \widetilde{\rho}_\phi(f) = \sup \{ f(|x|) - \rho_\phi(|x|) : \rho_\phi(|x|) < \infty \}
\]
\[
  \leq \sup \{ f(x) - \rho_\phi(x) : 0 \leq x \in \omega, \rho_\phi(x) < \infty \}.
\]

We shall need the following definition.

A linear functional \(f\) on \(1^*\) is said to be bounded for \(\rho_\phi\) (see [16], [17]) if there exists \(\gamma > 0\) such that
\[
  |f(x)| = \gamma (\rho_\phi(x) + 1) \quad \text{for } x \in 1^*.
\]
The collection of all bounded for \(\rho_\phi\) linear functionals on \(1^*\) will be denoted by \(1^\circ\).

The basic properties of \(\widetilde{\rho}_\phi\) are included in the following

**Theorem 2.5.** The conjugate \(\widetilde{\rho}_\phi\) of the modular \(\rho_\phi\) is a convex orthogonal additive modular on \(1^*\). Moreover, the following equality holds: \((1^*)^* = 1^\circ\).

**Proof.** Using [17, §4] and arguing as in the proof of [16, Theorem 38.2] we obtain that \(\widetilde{\rho}_\phi\) is a convex orthogonal additive modular on \(1^\circ\). To end the proof it suffices to show that \((1^*)^* = 1^\circ\). Indeed, let \(f \in (1^*)^*\) and \(\rho_\phi(x) < \infty\). Then \(\rho_\phi(x) \leq 1\) and there exists \(\gamma > 0\) such that \( |f(x)| \leq \gamma (\max (\rho_\phi(x), \| x \|_2) \leq \gamma (\rho_\phi(x) + 1) \leq \gamma (\rho_\phi(x) + 1), \) because \(\Phi(u) \leq \Phi(\phi(x)) \) for \(u \geq 0\). Thus \(f \in 1^\circ\); hence \((1^*)^* \subset 1^\circ\). Next, let \(f \in 1^\circ\) and let \(|x|_4 < 1\). Then \(\rho_\phi(x) \leq 1\), and hence \( |f(x)| \leq 2\gamma \) for some \(\gamma > 0\). This means that \(f \in (1^*)^*\), and thus \(1^\circ \subset (1^*)^*\). The proof is completed.

Thus by means of \(\widetilde{\rho}_\phi\) two modular norms can be defined on \((1^*)^*\) in a usual way (see [16], [17]):
\[
  \| f \|_{\widetilde{\rho}_\phi} = \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} (\widetilde{\rho}_\phi(\lambda f) + 1) \right\} \quad \text{(the first modular norm)}
\]
\[ \| f \|_{\mathcal{B}} = \inf \{ \lambda > 0 : \tilde{\psi}(f/\lambda) \leq 1 \} \] (the second modular norm).

3. Order Continuous Linear Functionals on \( 1^\ast \). We shall start this section with a description of the Köthe dual \((1^\ast)^\ast\) of \( 1^\ast \), that will be useful in obtaining a corresponding characterization of order continuous linear functional on \( 1^\ast \) (see [20, Proposition 1.9]).

Let us recall that the Köthe dual \( S^\ast \) of a sequence space \( S \) is the sequence space defined by (see [10, §30.1]):

\[ S^\ast = \left\{ y \in \omega : \sum_{i=1}^{\infty} |x(i)y(i)| < \infty \text{ for all } x \in S \right\}. \]

**Theorem 3.1.** The following equalities hold:

\[(1^\ast)^\ast = (h^\ast)^\ast = (h^\ast)^\ast = 1^\ast.\]

In particular, if \( \liminf_{u \to 0} \phi(u)/u > 0 \), then \((1^\ast)^\ast = 1^\ast\).

**Proof.** First, we shall show that \((1^\ast)^\ast = (h^\ast)^\ast = (h^\ast)^\ast = 1^\ast\). Since \((1^\ast)^\ast \subseteq (h^\ast)^\ast \subseteq (h^\ast)^\ast \subseteq (h^\ast)^\ast \), it suffices to show that \((h^\ast)^\ast \subseteq (1^\ast)^\ast \) and \((h^\ast)^\ast \subseteq (1^\ast)^\ast \). Indeed, let \( y \in (h^\ast)^\ast \), i.e., \( \sum_{i=1}^{\infty} |z(i)y(i)| < \infty \) for all \( z \in h^\ast \). Putting \( g_y(z) = \sum_{i=1}^{\infty} z(i)y(i) \) for \( z \in h^\ast \), by [20, Proposition 1.9] and Theorem 2.3 we get

\[ g_y \in (h^\ast)^\ast \subseteq (1^\ast)^\ast \subseteq (h^\ast)^\ast \subseteq (h^\ast)^\ast. \]

Therefore, we can put

\[ \| g_y \|_{\mathcal{B}} = \sup \left\{ \sum_{i=1}^{\infty} z(i)y(i) : z \in h^\ast, \| z \|_{\mathcal{B}} \leq 1 \right\}. \]

Let now \( x \in 1^\ast \) (resp. \( x \in h^\ast \), \( x = 0 \)). We shall show that \( \sum_{i=1}^{\infty} |x(i)y(i)| < \infty \). Since \( x \in 1^\ast \) and \( x^\omega \in h^\ast \) we get

\[ \frac{1}{\| x \|_{\mathcal{B}}} \sum_{i=1}^{\infty} |x(i)y(i)| = \frac{1}{\| x \|_{\mathcal{B}}} \sup_{z \in h^\ast} \sum_{i=1}^{\infty} |z(i)y(i)| \cdot \text{sign } y(i) \cdot y(i) \]

\[ \leq \sup \left\{ \sum_{i=1}^{\infty} z(i)y(i) : z \in h^\ast, \| z \|_{\mathcal{B}} \leq 1 \right\} = \| g_y \|_{\mathcal{B}} < \infty. \]

Hence \( y \in (1^\ast)^\ast \) (resp. \( y \in (h^\ast)^\ast \)), so that \((1^\ast)^\ast = (h^\ast)^\ast = (h^\ast)^\ast = 1^\ast\).

We have \((h^\ast)^\ast = (h^\ast)^\ast = (h^\ast, \tau_{\psi|h^\ast})^\ast \). It is well-known that by the mapping \( y \to g_y \) the space \((h^\ast)^\ast \) can be identified with \((h^\ast, \tau_{\psi|h^\ast})^\ast \) (see [20, Proposition 1.9]), and the space \( 1^\ast \) with \((h^\ast, \tau_{\psi|h^\ast})^\ast \) (see [12, Ch. II, §3, Theorem 2]). Thus \((h^\ast)^\ast = 1^\ast \), and since \( \bar{\psi} = \psi^\ast \sim \phi \), the proof is complete.

**Remark.** The equality \((1^\ast)^\ast = 1^\ast \) has been obtained by the author in [18] in a different way, using the so-called modular topology on \( 1^\ast \).

**Remark.** Assume now that \( \phi \) is an Orlicz function, not necessarily satisfying the condition: \( \phi(u)/u \to \infty \) as \( u \to \infty \). Let \( \psi \) be any Orlicz function such that \( \psi(u)/u = \phi(u) \) for \( 0 \leq u \leq 1 \), and \( \psi(u)/u \to \infty \) as \( u \to \infty \). Then in view of Theorem 3.1 we get \((1^\ast)^\ast = (\psi^\ast)^\sim = 1^\ast \). Thus, by Lemma 3.1 we get \((1^\ast)^\ast = 1^\ast \) for \( 0 < p \leq 1 \).

We are now able to give a characterization of order continuous linear functionals on \( 1^\ast \).

**Theorem 3.2.** Let \( f \) be a linear functional on \( 1^\ast \).

(a) The following statements are equivalent:
(1) $f$ is order continuous.

(2) There exists a unique $y \in \ell^*$ such

$$f(x) = f_0(x) + \sum_{i=1}^\infty x(i)y(i) \quad \text{for all } x \in \ell^*.$$

(b) If $f$ is order continuous, then the following equalities hold:

$$\begin{align*}
\|f\|_{\ell^*} &= \|f_0\|_{\ell^*} + \|y\|_{\ell^*}, \\
\|f\|_{\ell^*} &= \|f\|_{\ell^*} + \|y\|_{\ell^*}.
\end{align*}$$

PROOF. (a) It follows from [20, Proposition 1.9] and Theorem 3.1.

(b) By (a) we have $f(x) = \sum_{i=1}^\infty x(i)y(i)$ for some $y \in \ell^*$ and all $x \in \ell^*$.

First, we shall show that $\tilde{\rho}_\psi(f) = \rho_\psi(y)$. From the definition of $\psi$ we easily obtain that $\tilde{\rho}_\psi(f) \leq \rho_\psi(y)$.

To prove that $\tilde{\rho}_\psi(f) \leq \rho_\psi(y)$ let us note that there exists $0 \leq z \in \omega$ such that

$$\phi(z(i)) + \psi(\|y(i)\|) = |z(i)y(i)| \quad \text{for } i = 1, 2, \ldots.$$

Putting $x(i) = (\text{sign} y(i)) \cdot z(i)$ for $i = 1, 2, \ldots$, we get

$$\rho_\psi(y) = \sum_{i=1}^\infty \phi(\|y(i)\|) = \sup_x \left| \sum_{i=1}^\infty z(i)y(i) - \sum_{i=1}^\infty \phi(z(i)) \right| \leq \tilde{\rho}_\psi(f).$$

In turn, we shall show that $\|f\|_{\ell^*} \leq \|y\|_{\ell^*}$. We have $\|y\|_{\ell^*} = \sup \left| \sum_{i=1}^\infty z(i)y(i) \right| : x \in \ell^*$, $\rho_\psi(z) \leq 1$, and hence $\|f\|_{\ell^*} \leq \|y\|_{\ell^*}$. On the other hand, let $z \in \ell^*$ with $\rho_\psi(z) \leq 1$. Putting $x(i) = \text{sign} y(i) \cdot |z(i)| (i = 1, 2, \ldots)$, we have $p_\psi(x^{(k)}) = 0$ and $p_\psi(x^{(k)}) \leq \rho_\psi(z) \leq 1$. Thus

$$\left| \sum_{i=1}^\infty z(i)y(i) \right| = \sup_x \left| \sum_{i=1}^\infty z^{(k)}(i)y(i) \right| = \|f\|_{\ell^*}.$$ 

Thus $\|y\|_{\ell^*} \leq \|f\|_{\ell^*}$ and hence $\|f\|_{\ell^*} = \|y\|_{\ell^*}$.

Moreover, since $\tilde{\rho}_\psi(\lambda f) = \rho_\psi(\lambda y)$ for $\lambda > 0$, we get $\|f\|_{\ell^*} = \|y\|_{\ell^*}$.

Next, we shall show that $\|f\|_{\ell^*} \leq \|y\|_{\ell^*}$. To prove that $\|f\|_{\ell^*} \leq \|y\|_{\ell^*}$, let us assume that

$$x \in \ell^*, \quad p_\psi(x) \leq 1 \quad \text{and} \quad \|x\|_{\ell^*} \leq 1. \quad \text{Then } x \in \ell^*, \quad \text{and by the Hölder's inequality (see [11,§9]) we get} \quad |f(x)| \leq \|x\|_{\ell^*} \cdot \|y\|_{\ell^*} = \|y\|_{\ell^*},$$

because $\phi^* = \tilde{\phi}$. Thus $\|f\|_{\ell^*} \leq \|y\|_{\ell^*}$. To prove that $\|y\|_{\ell^*} \leq \|f\|_{\ell^*}$, let us note that (see [11, p. 135]):

$$\|f\|_{\ell^*} \leq \|y\|_{\ell^*}, \quad \text{and hence } \|f\|_{\ell^*} = \|y\|_{\ell^*}.$$
Let now \( z \in \ell^1 \) and \( \|z\|_1 \leq 1 \). Putting \( x(i) = (\text{sign} y(i)) \cdot |z(i)| \) \((i = 1, 2, \ldots)\) we have \( p_i(x^{\alpha}) = 0 \), \( \|x(n)\|_1 \leq \|z\|_1 \leq 1 \), and as above we get \( \|y\|_{\phi} \leq \|f\|_{\phi} \).

Finally, since \( \rho_\phi(f/\lambda) = \rho_\phi(y/\lambda) \) for \( \lambda > 0 \), we get \( \|f\|_{\phi} = \|y\|_{\phi} \).

(c) See [9, Ch. VI, §1, Theorem 1] and [14, Theorem 18.5].

**REMARK.** The general form of \( \phi \)-continuous (continuous with respect to the modular \( \rho_\phi \)) linear functionals on an Orlicz space \( L^\phi(a, b) \) defined by an Orlicz function satisfying conditions \( \phi(u)/u \to 0 \) as \( u \to 0 \) and \( \phi(u)/u \to \infty \) as \( u \to \infty \), has been found by W. Orlicz [19].

4. **Singular Linear Functionals on \( \ell^1 \).** In this section we assume that \( \phi \) does not satisfy the \( \Delta_2 \)-condition at 0, because otherwise \( (\ell^1)^*_\phi = \{0\} \).

The following lemma describes positive singular linear functionals on \( \ell^1 \).

**LEMMA 4.1.** Let \( \phi \) be a positive singular linear functional on \( \ell^1 \).

(a) For any \( \varepsilon > 0 \) there exists \( 0 < y \in \omega \) with \( \rho_\phi(y) < \varepsilon \) such that \( \|f\|_{\phi} = f(y) \).

(b) The following equalities hold:

\[
\rho_\phi(f) = \|f\|_{\phi}^\phi = \|f\|_{\phi} - \|f\|_{\phi}^* = \sup \{f(x) : 0 \leq x \in \ell^1, \rho_\phi(x) \leq 1\}.
\]

(c) There exists \( 0 < y \in \omega \) with \( \rho_\phi(y) < \infty \) such that

\[
\|f\|_{\phi} = f(y_A) \quad \text{for any subset } A \text{ of } N
\]

and

\[
\rho_\phi(y_A) = 1 \quad \text{for any subset } A \text{ of } N \text{ with } \|f\|_{\phi}^* = 0.
\]

**PROOF.** (a) Let \( \varepsilon > 0 \) be given. Since (see [26, Lemma 102.1])

\[
\|f\|_{\phi} = \sup \{f(x) : 0 \leq x \in \ell^1, \rho_\phi(x) \leq 1\},
\]

for every \( k \in N \) there exists \( 0 \leq z_k \in \ell^1 \) such that \( \rho_\phi(z_k) < 1 \) and \( \|z_k\|_{\phi} = f(z_k) + \frac{1}{k} \). Then \( \rho_\phi(z_k) < \infty \) and there exists a strictly increasing sequence of natural numbers \( (n_k) \) such that

\[
\rho_\phi(z_k - z_k^{(n_k)}) = \sum_{i=n_k}^{\infty} \phi(z_k(i)) < \frac{\varepsilon}{2^k}.
\]

Let \( x_k = z_k - z_k^{(n_k)} \) for \( k = 1, 2, \ldots \). Then in view of the axion (C) of completeness of the modular \( \rho_\phi \) there exists \( 0 \leq y \in \omega \) such that \( x_k \leq y \), for all \( k \in N \), and \( \rho_\phi(y) = \sum_{k=1}^{\infty} \rho_\phi(x_k) < \varepsilon \). But \( x_k^{(n_k)} \in \ell^1 \) for all \( k \in N \), so that

\[
\|z_k - z_k^{(n_k)}\|_{\phi} \leq f(x_k) + f(x_k^{(n_k)}) + \frac{1}{k} = f(y) + \frac{1}{k}.
\]

Since \( \varepsilon > 0 \) and \( k \) are arbitrary, we conclude that \( \|z_k - z_k^{(n_k)}\|_{\phi} \leq f(y) \).

(b) We have

\[
\|f\|_{\phi} \leq \|f\|_{\phi}^* = \sup \{f(x) : 0 \leq x \in \ell^1, \rho_\phi(x) \leq 1\}.
\]

To prove that \( \sup \{f(x) : 0 \leq x \in \ell^1, \rho_\phi(x) \leq 1\} \) assume that \( 0 \leq x \in \ell^1 \) and
Given $\eta > 0$, there exists $n \in \mathbb{N}$ such that $p_{\psi}(x - x^{(k)}) < \eta$. Then
\[
\| x - x^{(k)} \|_1 \leq 1 + p_{\psi}(x - x^{(k)}) \leq 1 + \eta
\]
and
\[
f(x) = f(x - x^{(k)}) + f(x^{(k)}) = f(x - x^{(k)}) = (1 + \eta) \| f \|_{L_\psi^1}.
\]

Hence $f(x) \leq \| f \|_{L_\psi^1}$, and thus we obtain
\[
\| f \|_{L_\psi^1} = \sup \{ f(x) : x \in \omega, \quad p_{\psi}(x) \leq 1 \}.
\]

Moreover, by (a) there exists $0 < y \in \omega$, with $p_{\psi}(y) \leq 1$, such that $\| f \|_{L_\psi^1} \leq f(y)$. Hence
\[
\frac{1}{n} = \sup \{ f(x) : x \in \omega, \quad p_{\psi}(x) \leq 1 \} = \sup \{ f(x) : x \in \omega, \quad p_{\psi}(x) < \infty \}
\]

Thus we proved that
\[
\| f \|_{L_\psi^1} = \sup \{ f(x) : x \in \omega, \quad p_{\psi}(x) \leq 1 \} = \sup \{ f(x) : x \in \omega, \quad p_{\psi}(x) < \infty \}.
\]

Finally, we shall show that $p_{\psi}(f) = \| f \|_{L_\psi^1}$. Indeed, by (a), for every $n \in \mathbb{N}$, there exists $0 \leq y_n \in \omega$, with $p_{\psi}(y_n) \leq \frac{1}{n}$, and such that $\| f \|_{L_\psi^1} \leq f(y_n)$. Hence
\[
p_{\psi}(f) = \sup \{ f(x) - p_{\psi}(x) : 0 \leq x \in \omega, \quad p_{\psi}(x) < \infty \}
\]
\[
= f(y_n) - p_{\psi}(y_n) = \| f \|_1 - \frac{1}{n}.
\]

Hence $p_{\psi}(f) \leq \| f \|_{L_\psi^1}$, and since
\[
p_{\psi}(f) = \sup \{ f(x) : 0 \leq x \in \omega, \quad p_{\psi}(x) < \infty \} = \| f \|_{L_\psi^1}
\]
we get $p_{\psi}(f) = \| f \|_{L_\psi^1}$. Thus the proof of (b) is completed.

(c) Let $A$ be a subset of $\mathbb{N}$, and let $0 \leq x \in \omega$ with $p_{\psi}(x) < \infty$ be given. Arguing as in (a) we obtain that there exists $0 \leq z \in \omega$ with $p_{\psi}(z) < \infty(k = 1, 2, \ldots)$ such that $\| f \|_{L_\psi^1} \leq f(z_k) + \frac{1}{k}$. Since $\| f \|_{L_\psi^1} = \sup \{ f(z) : 0 \leq z \in \omega, p_{\psi}(z) < \infty \}$ (see (b)), we have
\[
f \leq f(x) \leq f(z_k) + \frac{1}{k}.
\]
for all $k \in \mathbb{N}$, because $p_{\psi}(x \lor z_k) \leq p_{\psi}(x) + p_{\psi}(z_k) < \infty$. But $(x \lor z_k) \leq x \lor z_k$, so we get
\[
f(x) \leq f((x \lor z_k)_k) \leq f(z_k) + \frac{1}{k} (k = 1, 2, \ldots).
\]

Choose an increasing sequence of natural numbers $(m_k)$ such that $p_{\psi}(z_k - z^{(m_k)}_k) < \frac{1}{k}$, and let $x_k = z_k - z^{(m_k)}_k$. Then in view of the axiom (C) of completeness of $\psi$, there exists $0 \leq y \in \omega$ such that $x_k \leq y$ for all $k \in \mathbb{N}$, and $p_{\psi}(y) \leq 1$. Hence
\[
f(x_k) \leq f(z_k) + \frac{1}{k} = f(x) + \frac{1}{k}.
\]
Thus we obtain that $\|f_1\|_1 = f(y_1)$, because by (b),

$$\|f_1\|_1 = \sup\{f(x) : 0 \leq x \in \omega, \ p_\ell(x) < \infty\}.$$ 

Assume now that $\|f_1\|_1 = 0$. Given $\eta > 0$ we have $p_\ell(y_1/(p_\ell(y_1) + \eta)) < \infty$, and hence, by (b),

$$\|f_1\|_1 = f((y_1/(p_\ell(y_1) + \eta)).$$

Thus $\|f_1\|_1 = (p_\ell(y_1) + \eta)\|f_1\|_1$, so $p_\ell(y_1) = 1$, because $p_\ell(y_1) \leq p_\ell(y) \leq 1$. Thus the proof of (c) is completed.

**COROLLARY 4.2.** The space $((1^+)_1, \| \cdot \|_1)$ is an abstract $L$-space.

**PROOF.** By Theorem 2.4, $((1^+)_1, \| \cdot \|_1)$ is a Banach lattice. Arguing as in the proof of Lemma 2 of [2] we can show that $\|f_1 + f_2\|_1 = \|f_1\|_1 + \|f_2\|_1$ for any $f_1, f_2 \in ((1^+)_1)^*$, and this means that $(1^+)_1$ is an abstract $L$-space (see [23, Ch. II, §9]).

By $ba(N)$ we denote the family of all bounded real valued finitely additive set functions on $N$. It is known that $ba(N)$ is a vector lattice with the usual ordering: $\nu_1 \geq \nu_2$ iff $\nu_1(A) \geq \nu_2(A)$ for all $A \subseteq N$. Then $\nu = \nu^- - \nu^+$ and $|\nu| = \nu^+ + \nu^-$, where $\nu^+$ and $\nu^-$ denote the positive and the negative part of $\nu \in ba(N)$. Moreover $ba(N)$ is a Banach space under the norm $\|\nu\| = |\nu|(N)$ (see [6, Ch. III, 14, 1.7]).

For given $f \in ((1^+)_1)^*$ let us put $\nu_f(A) = \|f_1\|_1$ for any subset $A$ of $N$. Then by Corollary 4.2, $\nu_f \in (ba(N))^*$ and $\|\nu_f\| = |\nu_f|(N) = \|f\|_1$.

The following definition is justified by Lemma 4.1.

A $\nu \in ba(N)$ is said to be in class $B_\ell(N)$ if there exists $0 \leq y \in \omega$, with $p_\ell(y) < \infty$, such that $p_\ell(y_1) = 1$ for any subset $A$ of $N$ with $|\nu|(A) = 0$.

One can show that $B_\ell(N)$ is a Riesz subspace of $ba(N)$. In view of Lemma 4.1 we have the following

**LEMMA 4.3.** If $f \in ((1^+)_1)^*$, then $\nu_f \in (B_\ell(N))^*$.

Thus we can define a mapping $T : ((1^+)_1)^* \to (B_\ell(N))^*$ given by

$$T(f) = \nu_f \text{ for any } f \in ((1^+)_1)^*.$$ 

In view of Corollary 4.2 the mapping $T$ is additive.

For any $\nu \in (ba(N))^*$ we define a positive functional $I_\nu$ on $(1^+)_1$ by

$$I_\nu(x) = \inf\left\{\sum_{i=1}^s p_\ell(x_i) : N(A_s)\right\}$$

where the infimum is taken over all finite disjoint partitions $(A_s)_1^s$ of $N$.

By the same argument as in the proof of Lemma 5 of [2] we can prove that the functional $I_\nu$ is additive on $(1^+)_1$. Thus $I_\nu$ has a unique positive extension to a linear functional on $1^*$ (see [1, Lemma 3.1]). This extension (denoted again by $I_\nu$) is given by $I_\nu(x) = I_\nu(x^*) - I_\nu(x^-)$ for all $x \in 1^*$.

**LEMMA 4.4.** If $\nu \in (ba(N))^*$, then $I_\nu \in ((1^+)_1)^*$ and $\|I_\nu\|_1 = \nu(N)$.

**PROOF.** Since $I_\nu$ is positive on $1^*$, $I_\nu$ is order bounded. It is seen that $I_\nu(x) = 0$ for all $x \in h^*$, so $I_\nu \in ((1^+)_1)^*$. Moreover, $\|I_\nu(x)\| = \leq I_\nu(x^*) + I_\nu(x^-) = I_\nu(|x|) \leq p_\ell(x)\nu(N)$ for all $x \in 1^*$, so $\|I_\nu\|_1 \leq \nu(N)$.
Thus we can define a mapping $G: (B_1(N))^* \rightarrow ((1^*)_0)^*$ by

$$G(v) = I_v \quad \text{for any} \quad v \in (B_1(N))^*.$$ 

**THEOREM 4.5.** The following statements hold:

1. $(G \circ T)(f) = f$ for any $f \in ((1^*)_0)^*$, i.e.,
   $$f(x) = I_v(x) \quad \text{for all} \quad x \in 1^*.$$

2. $(T \circ G)(v) = v$ for any $v \in (B(N))^*$, i.e.,
   $$v(A) = \|I_v\|_{1^*}^* \quad \text{for any subset} \ A \text{ of} \ N.$$

**PROOF.** (1) Using Corollary 4.2 and Lemma 4.4, it suffices to repeat the arguments of the proof of Theorem 2 of [2].

(2) We first prove the case $A = N$. Since $v \in (B_1(N))^*$, there exists $0 \leq y \leq \omega$ such that $\rho_v(y) < \infty$ and $\rho_v(y_E) = 1$ for any subset $E$ of $N$ with $v(E) > 0$. Then for any finite disjoint partition $(E_k)_k$ of $N$ we have

$$\sum_{k=1}^\infty \rho_v(y_{E_k})v(E_k) = v(N),$$

so $I_v(y) = v(N)$. According to Lemma 4.1, we have $\|I_v\|_{1^*} = I_v(y) - v(N)$. Moreover, we have $I_v(x) \leq \rho_v(x)v(N)$ for all $0 \leq x \leq 1^*$. Hence $\|I_v\|_{1^*} \leq v(N)$, so $\|I_v\|_{1^*} = v(N)$. Assume now that $A$ is a fixed subset of $N$, and let $v_1(B) = v(A \cap B)$ for any $B \subset N$. One can easily show that $I_v_1 = (I_v)_A$. Hence, by the above, we get $\|I_v_1\|_{1^*} = \|I_{v_1}\|_{1^*} = v_1(N) = v(A)$, and the proof is completed.

By Theorem 4.5 the mapping $G$ is additive, because $T$ is additive. Thus $T$ and $G$ have unique positive extensions to linear mappings $\hat{T}: (1^*)_0 \rightarrow B_1(N)$ and $\hat{G}: B_1(N) \rightarrow (1^*)_0$ (see [1, Lemma 3.1]) given by

$$\hat{T}(f) = v_f - v_f \quad \text{and} \quad \hat{G}(v) = I_v - I_v.$$

Let us put: $v_f = v_f - v_f$ and $I_v = I_v - I_v$. For any $v \in B_1(N)$ we shall write

$$\int x dv = I_v(x) \quad \text{for all} \quad x \in 1^*.$$

**THEOREM 4.6.** (see [2, Theorem 4]). The mapping $\hat{T}: (1^*)_0 \rightarrow B_1(N)$ is a Riesz isomorphism.

**PROOF.** In view of Theorem 4.5, we get $(\hat{G} \circ \hat{T})(f) = f$, for any $f \in (1^*)_0$, and $(\hat{T} \circ \hat{G})(v) = v$, for any $v \in B_1(N)$. Thus $\hat{T}$ is a Riesz isomorphism, because $\hat{T}$ is positive (see [14, Theorem 18.5]).

The final result of this section gives a characterization of singular linear functionals on $1^*$.

**THEOREM 4.7.** Let $f$ be a linear functional on $1^*$.

(a) The following statements are equivalent:

1. $f$ is singular.
2. There exists a unique $v \in B_1(N)$ such that
   $$f(x) = \int x dv \quad \text{for all} \quad x \in 1^*.$$

(b) If $f$ is singular, then the following equalities hold:

$$\rho_f(f) = \|f\|_{1^*} - \|f\|_{1^*} = \|f\|_{1^*} - \|f\|_{1^*} = \|v\|_{1^*}.$$

**PROOF.** (a) See the proof of Theorem 4.6.
According to Theorem 4.6, we get \( v_{ll}(N) = \vert v \vert (N) \). Thus, in view of Lemma 4.1, we get

\[
\overline{\rho}_g(f) = \rho_g(|f|) - I |f| \overline{\rho}_g - I |f| \overline{\rho}_g = ||f||_g - ||f||_g - \vert v \vert (N).
\]

Moreover, since \( \overline{\rho}_g(\lambda f) = \rho_g(\lambda |f|) = \lambda \overline{\rho}_g(f) \) for \( \lambda > 0 \) (see Lemma 4.1), we obtain that \( ||f||_g = \rho_g(f) \) and \( ||f||_g = \rho_g(f) \). Since the norms which occur in our theorem are Riesz norms the proof is complete.

Since \((1^*, \| \cdot \|_1)\) is an abstract L-space (see Corollary 4.2), by Theorems 4.6 and 4.7, we obtain that \( B_q(N) \) is also an abstract L-space.

5. The General Form of Continuous Linear Functionals on \( 1^* \). We are now in position to give a desired characterization of the dual space \((1^*)^*\).

**Theorem 5.1.** Let \( f \) be a linear functional on \( 1^* \).

(a) The following statements are equivalent:

1. \( f \) is \( \tau_0 \)-continuous.
2. \( f \) is order bounded.
3. There exist unique \( y \in 1^* \) and \( v \in B_q(N) \) such that

\[
f(x) = \sum_{i=1}^n x(i) y(i) + \int x dv \quad \text{for all } x \in 1^*.
\]

(b) If \( f \) is \( \tau_0 \)-continuous, then the following equalities hold:

\[
\overline{\rho}_g(f) = \rho_g(y) + \vert v \vert (N),
\]

\[
\| f \|_g = \| f \|_{\overline{\rho}_g} + \| y \|_g + \vert v \vert (N).
\]

(c) The space \( h^* \) is an M-ideal of \((1^*, \rho_0 \| \cdot \|_0)\).

**Proof.** (a) It follows from Theorem 2.4, Theorem 3.2 and Theorem 4.7.

(b) By Theorem 2.4, we have \( f = f - f + f \), and it is known that \( |f|_0 = |f|_0 \), \( |f|_0 = |f|_0 \), and \( |f|_0 = |f|_0 = 0 \). Since the conjugate modular \( \overline{\rho}_g \) is orthogonal additive on \((1^*)^*\), by Theorem 3.2 and Theorem 4.7, we get \( \overline{\rho}_g(f) = \rho_g(f) + \overline{\rho}_g(f) = \rho_g(y) + \vert v \vert (N) \).

We shall now show that \( \| f \|_g = \| y \|_g + \| v \|_g \). Indeed, let \( \varepsilon > 0 \) be given. Then there exists \( 0 \leq \varepsilon \in 1^* \) with \( p_0(\varepsilon) < 1 \), \( \rho_0(\varepsilon) < 1 \), such that

\[
\| f \|_g = \| f \|_0 + \| f \|_0 < \| f \|_0 (\varepsilon) + \varepsilon.
\]

Moreover, in view of Lemma 4.1 there exists \( 0 \leq \varepsilon \in 1^* \) such that

\[
\| f \|_0 + \| f \|_0 = \| f \|_0 (\varepsilon).
\]

Let \( z = x \vee y \). Then \( \rho_g(z) \leq \rho_g(x) + \rho_g(y) \leq 1 \). Moreover, since \( p_0(\varepsilon) < 1 \), we have \( \rho_0(\varepsilon) < \infty \). Hence \( \rho_0(\varepsilon) < \infty \), so \( p_0(\varepsilon) \leq 1 \). Thus

\[
\| f \|_0 + \| f \|_0 = \| f \|_0 (\varepsilon) + \| f \|_0 (\varepsilon) + \varepsilon = \| f \|_0 (\varepsilon) + \varepsilon.
\]
Hence \( \|f\|_{\psi} + \|f\|_{\psi} = \|f\|_{\psi} \), and, according to Theorem 3.2 and Theorem 4.7, we obtain \( \|f\|_{\psi} = \|y\|_{\\psi} + |v| \) (N). Finally, since \( \widetilde{p}_{\alpha}(\lambda f_{\alpha}) = \rho_{\alpha}(\lambda y) \) and \( \tilde{p}_{\alpha}(\lambda f_{\alpha}) = \rho_{\alpha}(\lambda y) |v| \) (N) for \( \lambda > 0 \), we easily obtain that \( \|f\|_{\psi} = \|y\|_{\\psi} + |v| \) (N).

(c) It is well known that \((h^*)^0 = (1^*)^0\) (see [26, Theorem 88.10]), where \((h^*)^0\) denotes the annihilator of \(h^*\) in \((1^*)^0\). Therefore, from (b) it follows that \((h^*)^0\) is an \(L\)-summand of \((1^*)^0, \|\\cdot \|_{\psi}\) (see [3, Definition 1.1]). According to [3, Definition 2.1] it means that \(h^*\) is an \(M\)-ideal of \((1^*, \|\\cdot \|_{\psi})\).

REMARK. For a convex Orlicz function \(\phi\) the equality \(\|f\|_{\psi} = \|f\|_{\psi}\) has been proved by W. A. Luxemburg and A. C. Zaanen [12, Theorem 5].

As an application of Theorem 5.1 we obtain that continuous linear functionals on \(h^*\) have the unique norm preserving extension to \(1^*\).

COROLLARY 5.3. (see [21, Proposition 3]). Let \(g\) be a \(\tau_{\psi, \|\\cdot \|_{\psi}}\)-continuous linear functional on \(h^*\). Then there exists a unique \(\tau_{\psi, \|\\cdot \|_{\psi}}\)-continuous linear functional \(f\) on \(1^*\) such that \(f(x) = g(x)\) for all \(x \in h^*\), and \(\|f\|_{\psi} = \|g\|_{\psi}\), where

\[
\|g\|_{\psi} = \sup\{ |g(x)| : x \in h^*, \|x\|_{\psi} \leq 1 \}.
\]

PROOF. Since \((h^*, \tau_{\psi, \|\\cdot \|_{\psi}})^* = (h^*)^0 = (1^*)^0\) (see [1, Theorem 16.9]), according to [20, Proposition 1.9] and Theorem 3.1 there exists a unique \(y \in 1^*\) such that \(g(x) = \sum_{i=1}^{\infty} x(i)y(i)\) for all \(x \in h^*\). Let us put
\[
f(x) = \sum_{i=1}^{\infty} x(i)y(i) \quad \text{for all} \quad x \in 1^*.
\]
Then \(f(x) = g(x)\) for \(x \in h^*\), and, according to Theorem 3.2, \(f\) is order continuous and \(\|f\|_{\psi} = \|y\|_{\psi}\). Now we shall show that \(\|g\|_{\psi} = \|f\|_{\psi}\). Indeed, we have \(\|g\|_{\psi} \leq \|f\|_{\psi}\). Let \(x \in 1^*\) with \(p_{\alpha}(x) \leq 1\), \(\|x\|_{\psi} \leq 1\). Then
\[
\left| \sum_{i=1}^{\infty} x(i)y(i) \right| \leq \sup_{n} \left| \sum_{i=1}^{n} x(i)y(i) \right| = \sup_{n} \sum_{i=1}^{n} |x(i)y(i)| \cdot \text{sign} \ y(i) \cdot y(i) \leq \|g\|_{\psi}.
\]
Hence \(\|f\|_{\psi} \leq \|g\|_{\psi}\), and we are done.

Now assume that \(\tilde{f}\) is another such extension of \(g\), and let \(F = \tilde{f} - f\). Then \(F\) is singular on \(1^*\) and \(\tilde{f} = f + F\). Hence, by Theorem 2.4, we have \(f = \tilde{f}\) and \(F = \tilde{f}\). Therefore, in view of Theorem 5.1, we have \(\|f\|_{\psi} = \|\tilde{f}\|_{\psi} = \|1^*\|_{\psi} + \|F\|_{\psi}\). Since \(\|\tilde{f}\|_{\psi} = \|g\|_{\psi} - \|y\|_{\psi}\), we obtain that \(F = 0\), so \(\tilde{f} = f\). Thus the proof is completed.

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