ON k -IDEALS OF SEMIRINGS

M.K. SEN

Department of Pure Mathematics
Calcutta University
35, Ballygunge Circular Road
Calcutta - 700 019 India

and

M.R. ADHIKARI

Department of Mathematics
University of Burdwan
Golapbag, Burdwan
West Bengal, India

(Received December 21, 1990)

ABSTRACT. Certain types of ring congruences on an additive inverse semiring are characterized with the help of full k -ideals. It is also shown that the set of all full k -ideals of an additively inverse semiring in which addition is commutative forms a complete lattice which is also modular.

KEY WORDS AND PHRASES. Semiring, inverse semiring, k -ideals and ring congruence.

1991 AMS SUBJECT CLASSIFICATION CODE. 16A-78.

1. **PRELIMINARIES.** A semiring is a system consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that

- (i) S together with addition is a semigroup;
- (ii) S together with multiplication is a semigroup; and
- (iii) $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a, b, c \in S$.

A semiring S is said to be additively commutative if $a+b = b+a$ for all $a, b \in S$. A left (right) ideal of a semiring S is non-empty subset I of S such that

- i) $a+b \in I$ for all $a, b \in I$; and
- ii) $ra \in I (ar \in I)$ for all $r \in S$ and $a \in I$.

An ideal of a semiring S is a non-empty subset I of S such that I is both a left and right ideal of S .

Henriksen [1] defined a more restricted class of ideals in a semiring, which he called k -ideals.

A left k -ideal I of a semiring S is a left ideal such that if $a \in I$ and $x \in S$ and if either $a+x \in I$ or $x+a \in I$, then $x \in I$.

Right k -ideal of a semiring is defined dually. A non-empty subset I of a semiring S is called a k -ideal if it is both a left k -ideal and a right k -ideal.

A semiring S is said to be additively regular if for each $a \in S$, there exists an element $b \in S$ such that $a = a+b+a$. If in addition, the element b is unique and satisfies $b = b+a+b$, then S is

called an additively inverse semiring. In an additively inverse semiring the unique inverse b of an element a is usually denoted by a' . Karvellas [2] proved the following result:

Let S be an additively inverse semiring. Then

- i) $x = (x')'$, $(x + y)' = y' + x'$, $(xy)' = x'y' = y'x$ and $xy = x'y'$ for all $x, y \in S$.
- ii) $E^+ = \{x \in S : x + x = x\}$ is an additively commutative semilattice and an ideal of S .

2. FULL k -IDEALS. In this section S denotes an additively inverse semiring in which addition is commutative and E^+ denotes the set of all additive idempotents of S .

A left k -ideal A of S is said to be full if $E^+ \subseteq A$. A right k -ideal of S is defined dually.

A non-empty subset I of S is called a full k -ideal if it is both left and a right full k -ideal.

EXAMPLE 1. In a ring every ring ideal is a full k -ideal.

EXAMPLE 2. In a distributive lattice with more than two elements, a proper ideal is a k -ideal but not a full k -ideal.

EXAMPLE 3. $Z \times Z^p = \{(a, b) : a, b \text{ are integers and } b > 0\}$. Define

$$(a, b) + (c, d) = (a + c, \text{l.c.m. of } b, d) \text{ and } (a, b)(c, d) = (ac, \text{h.c.f. of } b, d).$$

Then $Z \times Z^p$ becomes an additively inverse semiring in which addition is commutative.

Let $A = \{(a, b) \in Z \times Z^p : a = 0, b \in Z^p\}$. Then A is a full k -ideal of $Z \times Z^p$.

LEMMA 2.1. Every k -ideal of S is an additively inverse subsemiring of S .

PROOF. Let I be a k -ideal of S . Clearly I is a subsemiring of S . Let $a \in I$. Then

$$a + (a' + a) = a \in I.$$

Since I is a k -ideal, it follows $a' + a \in I$. Again this implies that $a' \in I$. Hence the lemma.

LEMMA 2.2. Let A be an ideal of S . Then

$$\overline{A} = \{a \in S : a + x \in A \text{ for some } x \in A\} \text{ is a } k\text{-ideal of } S.$$

PROOF. Let $a, b \in \overline{A}$. The $a + x, b + y \in A$ for some $x, y \in A$. Now

$$a + x + b + y = (a + b) + (x + y) \in A.$$

As $x + y \in A, a + b \in \overline{A}$. Next let $r \in S, ra + rx = r(a + x) \in A$.

As $rx \in A, ra \in \overline{A}$. Similarly, $ar \in \overline{A}$. As a result $\overline{A}$ is an ideal of S . Next, let c and $c + d \in \overline{A}$. Then there exists x and y in A such that $c + x \in A$ and $c + d + y \in A$.

Now

$$d + (c + x + y) = (c + d + y) + x \in A \text{ and } c + x + y \in A.$$

Hence $d \in \overline{A}$ and $\overline{A}$ is a k -ideal of S . Since $a + a' \in A$ for all $a \in A$, it follows that $A \subseteq \overline{A}$.

COROLLARY. Let A be an ideal of S . Then $\overline{A} = A$ iff A is a k -ideal.

LEMMA 2.3. Let A and B be two full k -ideals of S , then $\overline{A + B}$ is a full k -ideal of S such that

$$A \subseteq \overline{A + B} \text{ and } B \subseteq \overline{A + B}.$$

PROOF. It can be shown that $A + B$ is an ideal of S . Then from Lemma 2.2, we find $\overline{A + B}$ is a k -ideal and $A + B \subseteq \overline{A + B}$. Now $E^+ \subseteq A, B$. Hence $E^+ \subseteq A + B \subseteq \overline{A + B}$. This implies that $\overline{A + B}$ is a full k -ideal. Let $a \in A$. Then

$$a = a + a' + a = a + (a' + a) \in A + B \text{ as } a' + a \in E^+ \subseteq B.$$

Hence $A \subseteq \overline{A + B}$ and similarly $B \subseteq \overline{A + B}$.

THEOREM 2.4. If $I(S)$ denotes the set of all full k -ideals of S , then $I(S)$ is a complete lattice which is also modular.

PROOF. We first note that $I(S)$ is a partially ordered set with respect to usual set inclusion. Let $A, B \in I(S)$. Then $A \cap B \in I(S)$ and from Lemma 2.3, $\overline{A+B} \in I(S)$. Define $A \wedge B = A \cap B$ and $A \vee B = \overline{A+B}$. Let $C \in I(S)$ such that $A, B \subseteq C$. Then $A+B \subseteq C$ and $\overline{A+B} \subseteq \overline{C}$. But $\overline{C} = C$. Hence $\overline{A+B} \subseteq C$. As a result $\overline{A+B}$ is the l.u.b. of A, B . Thus we find that $I(S)$ is a lattice. Now E^+ is an ideal of S . Hence $\overline{E^+} \in I(S)$ and also $S \in I(S)$; consequently $I(S)$ is a complete lattice. Next suppose that $A, B, C \in I(S)$ such that

$$A \wedge B = A \wedge C \text{ and } A \vee B = A \vee C \text{ and } B \subseteq C.$$

Let $x \in C$. Then $x \in A \vee C = A \vee B = \overline{A+B}$. Hence there exists $a+b \in A+B$ such that $x+a+b = a_1+b_1$ for some $a_1 \in A, b_1 \in B$.

Then

$$x+a+a'+b = a_1+b_1+a'.$$

Now $x \in C, a+a' \in C$ and $b \in B \subseteq C$. Hence $a_1+b_1+a' \in C$. But $b_1 \in C$. Consequently, $a_1+a' \in C \cap A = C \cap B$. Hence $a_1+a' \in B$. So from $x+a+b = a_1+b_1$ we find that $x+a+a'+b = a_1+a'+b \in B$. But $(a+a')+b \in B$ and B is a k -ideal. Hence $x \in B$ and $B = C$. This proves that $I(S)$ is a modular lattice.

3. RING CONGRUENCES.

A congruence ρ on a semiring S is called a ring congruence if the quotient semiring S/ρ is a ring.

In this section we assume S is an additively inverse semiring in which addition is commutative. We want to characterize those ring congruences on S such that $-(a\rho) = a'\rho$ where a' denotes the inverse of a in S and $-(a\rho)$ denotes the additive inverse of $a\rho$ in the ring S/ρ .

THEOREM 3.1. Let A be a full k -ideal of S . Then the relation

$$\rho_A = \{(a, b) \in S \times S : a+b' \in A\} \text{ is a ring congruence on } S \text{ such that } -(a\rho_A) = a'\rho_A.$$

PROOF. Since $a+a' \in E^+ \subseteq A$ for all $a \in S$, it follows that ρ_A is reflexive. Let $a+b' \in A$. Now from Lemma 2.1, we find that $(a+b')' \in A$. Then $b+a' = (b')' + a' = (a+b')' \in A$. Hence ρ_A is symmetric. Let $a+b' \in A$ and $b+c' \in A$. Then $a+b+b'+c' \in A$. Also $b+b' \in E^+ \subseteq A$. Since A is a k -ideal, we find that $a+c' \in A$. Hence ρ_A is an equivalence relation. Let $(a, b) \in \rho_A$ and $c \in S$. Then $a+b' \in A$. Since

$$(c+a) + (c+b)' = c+a+b'+c' = (a+b') + (c+c') \in A, \quad ca + (cb)' = ca+cb' = c(a+b') \in A,$$

$$ac + (bc)' = ac+b'c = (a+b')c \in A,$$

it follows that ρ_A is a congruence on S . So we obtain the quotient semiring where addition and multiplication are defined by

$$a\rho_A + b\rho_A = (a+b)\rho_A \text{ and } (a\rho_A)(b\rho_A) = (ab)\rho_A.$$

Now

$$a\rho_A + b\rho_A = (a+b)\rho_A = (b+a)\rho_A = b\rho_A + a\rho_A.$$

Let $e \in E^+$ and $a \in S$. Now $(e+a) + a' = e + (a+a') \in E^+$.

We find that $(e+a)\rho_A = a\rho_A$. Then $e\rho_A + a\rho_A = a\rho_A$.

Also

$$a\rho_A + a'\rho_A = (a + a')\rho_a = e\rho_A.$$

Hence $e\rho_A$ is the zero element and $a'\rho_A$ is the negative element of $a\rho_A$ in the ring S/ρ_A .

THEOREM 3.2. Let ρ be a congruence on S such that S/ρ is a ring and $-(a\rho) = a'\rho$. Then there exists a full k -ideal A of S such that $\rho_A = \rho$.

PROOF. Let $A = \{a \in S : (a, e) \in \rho \text{ for some } e \in E^+\}$. Since ρ is reflexive, it follows that $E^+ \subseteq A$. Then $A \neq \phi$, since $E^+ \neq \phi$. Let $a, b \in A$. Then there exist $e, f \in E^+$ such that $(a, e) \in \rho$ and $(b, f) \in \rho$. Then $(a + b, e + f) \in \rho$. But $e + f \in E^+$. Hence $a + b \in A$. Again for any $r \in S$, $(ra, re) \in \rho$ and $(ar, er) \in \rho$. But re and $er \in E^+$. Hence A is an ideal of S .

Let $a + b \in A$ and $b \in A$. Then there exist $e, f \in E^+$ such that $(a + b, f) \in \rho$ and $(b, e) \in \rho$. Hence $f\rho = (a + b)\rho = a\rho + b\rho = a\rho + e\rho$. But $f\rho$ and $e\rho$ are additive idempotents in the ring S/ρ . Hence $e\rho = f\rho$ is the zero element of S/ρ . As a result, $a\rho$ is the zero element of S/ρ . Then $a\rho = e\rho$. This implies $a \in A$. So we find that A is a full k -ideal of S . Consider now the congruences ρ_A and ρ . Let $(a, b) \in \rho$. Then $(a + b', b + b') \in \rho$. But $b + b' \in E^+$. Hence $a + b' \in A$ and $(a, b) \in \rho_A$. Conversely suppose that $(a, b) \in \rho_A$. Then $a + b' \in A$. Hence $(a + b', e) \in \rho$ for some $e \in E^+$. As a result, $e\rho = a\rho + b'\rho = a\rho - b\rho$ holds in the ring S/ρ . But $e\rho$ is the zero element of S/ρ . Consequently $a\rho = b\rho$. This shows that $(a, b) \in \rho$ and hence $\rho_A = \rho$.

REFERENCES

1. HENRIKSEN, M., Ideals in semirings with commutative addition. Amer. Math. Soc. Notices, **6** (1958), 321.
2. KARVELLAS, PAUL H., Inverse semirings, J. Austral. Math. Soc. **18** (1974), 277-288.