MEASURES ON COALLOCATION AND NORMAL LATTICES

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ABSTRACT

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices of subsets of a nonempty set $X$. Suppose $\mathcal{L}_3$ coallocates $\mathcal{L}_1$ and $\mathcal{L}_3$ is a subset of $\mathcal{L}_2$. We show that any $\mathcal{L}_1$-regular finitely additive measure on the algebra generated by $\mathcal{L}_1$ can be uniquely extended to an $\mathcal{L}_3$-regular measure on the algebra generated by $\mathcal{L}_2$. The case when $\mathcal{L}_3$ is not necessary contained in $\mathcal{L}_2$, as well as the measure enlargement problem are considered. Furthermore, some discussions on normal lattices and separation of lattices are also given.

KEY WORDS: lattices, normal lattices, coallocation lattices, semi-separated lattices, regular finitely additive measures, $\sigma$-smooth measures, measure extension, measure enlargement.


1. INTRODUCTION

Let $X$ be an arbitrary set and $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices of subsets of $X$. If $\mathcal{L}_1 \subset \mathcal{L}_2$, and if $\mathcal{L}_2$ coallocates $\mathcal{L}_1$, then any $\mathcal{L}_1$-regular finitely additive measure on the algebra generated by $\mathcal{L}_1$ can be uniquely extended to an $\mathcal{L}_2$-regular measure on the algebra generated by $\mathcal{L}_2$. This situation has been investigated by J. Camacho in [2]. We extend his results in several directions in this paper. We will consider the case where $\mathcal{L}_1$ is not necessary contained in $\mathcal{L}_2$ (see Theorem 3.1) and show that under suitable conditions any $\mu \in \mathcal{M}_R(\mathcal{L}_1)$ (see below for definitions) gives rise to a $\nu \in \mathcal{M}_R(\mathcal{L}_2)$. We will also
consider besides measure extension problems, measure enlargement problems (see e.g. Theorem 3.3) and will finally apply these results to the case of a single lattice \( \mathcal{L} \), thereby extending results of M. Szeto [8] for measures on normal lattices.

We begin by giving some standard lattice and measure theoretic background in Section 2. Our notation and terminology is consistent with [1,4,6,7,9]. In Section 3, we consider the general coalllocation theorem and a variety of consequences of it. Section 4 is devoted to a more detailed discussion of normal lattices and to separation of lattices. This work extends to some extent that of G. Eid [3].

2. BACKGROUND AND TERMINOLOGY

In this section, we summarize some lattice and measure theoretic notions and notations. This is all fairly standard and as previously mentioned is consistent with standard references.

Definition 2.1

Let \( X \) be a nonempty set and \( \mathcal{P}(X) \) is the power set of \( X \). A lattice \( \mathcal{L} \) is a collection of subsets of \( X \), which is closed under finite unions and finite intersections, and \( \emptyset, X \in \mathcal{L} \). Let

\[
\mathcal{L}' = \{ L' : L \in \mathcal{L} \}
\]

where \( L' \) denotes the complement of \( L \). \( \mathcal{L}' \) is a lattice if \( \mathcal{L} \) is.

Definition 2.2

Let \( \mathcal{L}, \mathcal{L}_1 \), and \( \mathcal{L}_2 \) be any lattices of subsets of \( X \).

1. \( \mathcal{L} \) is \( \delta \) if it is closed under countable intersections.

2. \( \mathcal{L} \) is a complement generated (c.g.) lattice if

\[
\forall L \in \mathcal{L}, \exists L_1, L_2, \ldots \in \mathcal{L} \text{ such that } L = \bigcap_{n=1}^{\infty} L_n'.
\]

3. \( \mathcal{L} \) is a normal lattice if

\[
\forall L_1, L_2 \in \mathcal{L}, L_1 \cap L_2 = \emptyset \implies \exists L_1', L_2' \in \mathcal{L} \text{ s.t. } L_1 \subseteq L_1', L_2 \subseteq L_2', L_1' \cap L_2' = \emptyset.
\]

4. \( \mathcal{L} \) is a countably paracompact (c.p.) lattice if

\[
\forall L_1, L_2, \ldots \in \mathcal{L}, L_1 \supseteq L_2 \supseteq \ldots, \lim_{n \to \infty} L_n = \emptyset (L_n \uparrow \emptyset) \implies \exists L_1', L_1, \ldots \in \mathcal{L} \text{ s.t. } \forall n, L_n \subseteq L_1' \text{ and } L_n \uparrow \emptyset.
\]

5. \( \mathcal{L}_2 \) is \( \mathcal{L}_1 \)-countably-paracompact (\( \mathcal{L}_1 \)-c.c.p.) if
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∀ B₁, B₂, ..., ∈ ℳ₂, B₁ ⊇ B₂ ⊇ ..., Bₙ ⊇ ∅ ⇒

∃ A₁, A₂, ..., ∈ ℳ₁ s.t. ∀ n, Bₙ ⊇ Aₙ, and Aₙ ⊇ ∅.

(6) ℳₙ semi-separates ℳₙ if

∀ A ∈ ℳ₁, B ∈ ℳ₂, A ∩ B = ∅ ⇒ ∃ L₁ ∈ ℳ₁, s.t. B ⊇ L₁ and A ∩ L₁ = ∅.

(7) ℳₙ separates ℳₙ if

∀ L₁, L₂ ∈ ℳ₁, L₁ ∩ L₂ = ∅ ⇒

∃ L₁, L₂ ∈ ℳ₁, s.t. L₂ ⊇ L₁, L₂ ⊇ L₁, and L₁ ∩ L₂ = ∅.

(8) ℳₙ coseparates ℳₙ if

∀ L₁, L₂ ∈ ℳ₁, L₁ ∩ L₂ = ∅ ⇒

∃ L₁, L₂ ∈ ℳ₁, s.t. L₂ ⊇ L₁, L₂ ⊇ L₁, and L₁ ∩ L₂ = ∅.

(9) ℳₙ coallocates ℳₙ if

∀ L₁ ∈ ℳ₁ s.t. L₁ ⊇ L₂ ⊇ L₂', where L₁, L₂ ∈ ℳ₁ ⇒

∃ L₁, L₂ ∈ ℳ₁ s.t. L₁ ⊇ L₂', L₁ ⊇ L₂, L₁ = L₂ ∪ L₂'.

Definition 2.3

A finitely additive (f.a.) measure µ is a finite nonnegative function defined on the algebra A(ℳ) generated by ℳ, such that

1) ∀ A ∈ A(ℳ), µ(A) ≥ 0, (2) µ(∅) = 0, and (3) [finite additivity] ∀ A, B ∈ A(ℳ),

A ∩ B = ∅ ⇒ µ(A ∪ B) = µ(A) + µ(B).

A 0-1 measure µ is a two-valued finitely additive measure taking value either 0 or 1.

Usually, we simply refer µ to as a measure on a lattice ℳ to mean that µ is a finitely additive measure defined on the algebra A(ℳ).

A f.a. measure µ defined on the algebra A(ℳ) is

1) ℳ-regular iff ∀ A ∈ A(ℳ), µ(A) = sup { µ(L₁) : L₁ ⊇ A, L₁ ∈ ℳ }.

Or, equivalently, µ(A) = inf { µ(L₁) : L₁ ⊇ A, L₁ ∈ ℳ }.

2) σ-smooth on A(ℳ), iff

∀ A₁, A₂, ..., ∈ A(ℳ), A₁ ⊇ A₂ ⊇ ..., (Aₙ ∩ ∅) ⇒ µ(Aₙ) → 0 as n→∞

3) σ-smooth on ℳ, iff

∀ L₁, L₂, ..., ∈ ℳ, L₁ ⊇ L₂ ⊇ ..., (Lₙ ∩ ∅) ⇒ µ(Lₙ) → 0 as n→∞

The following notations for the collections of measures on A(ℳ) will be used throughout:

M(ℳ) = { µ : µ f.a. measure on A(ℳ) }

M₆(ℳ) = { µ ∈ M(ℳ) : µ ℳ-regular }

Mσ(ℳ) = { µ ∈ M(ℳ) : µ σ-smooth on A(ℳ) }

Mₐ(ℳ) = { µ ∈ M(ℳ) : µ σ-smooth on ℳ }
$M_\sigma^0(\mathcal{F}) = \{ \mu \in M(\mathcal{F}) : \mu$ \sigma-smooth on $A(\mathcal{F})$ and $\mathcal{F}$-regular $\}$

Similarly, we also define $I(\mathcal{F})$, $I_R(\mathcal{F})$, $I^{0}(\mathcal{F})$, $I_\sigma(\mathcal{F})$, and $I_R^0(\mathcal{F})$ for non-trivial 0-1 measures.

If $\mu$ is $\mathcal{F}$-regular, then $\sigma$-smoothness on $\mathcal{F}$ implies $\sigma$-smoothness on $A(\mathcal{F})$. Thus, $M_\sigma^0(\mathcal{F}) = M_R(\mathcal{F}) \cap M_\sigma(\mathcal{F}) = M_R(\mathcal{F}) \cap M^0(\mathcal{F})$.

Since $A(\mathcal{F}') = A(\mathcal{F})$, we have $M(\mathcal{F}') = M(\mathcal{F})$ and $I(\mathcal{F}') = I(\mathcal{F})$.

Furthermore, $\mu$ is $\sigma$-smooth on $A(\mathcal{F})$ ($\mu \in M^0(\mathcal{F})$) iff $\mu$ is countably additive.

Let $\mu_1, \mu_2 \in M(\mathcal{F})$. Define

1. $\mu_1 \leq \mu_2$ if $\forall A \in A(\mathcal{F})$, $\mu_1(A) \leq \mu_2(A)$
2. $\mu_1 \leq \mu_2$ on $\mathcal{F}$, if $\forall L \in \mathcal{F}$, $\mu_1(L) \leq \mu_2(L)$
3. $\mu_1 \leq \mu_2$ on $\mathcal{F}'$, if $\forall L \in \mathcal{F}$, $\mu_1(L') \leq \mu_2(L')$

**Definition 2.4**

Suppose $\mathcal{F}_1 \subset \mathcal{F}_2$ are lattices of subsets of $X$ such that $\mu_1 \in M(\mathcal{F}_1)$ and $\mu_2 \in M(\mathcal{F}_2)$. Denote $\mu_2 | \mathcal{F}_1$ (or simply $\mu_2 |$) to mean the restriction of $\mu_2$ to $A(\mathcal{F}_1)$.

If $\mu_1 = \mu_2 |$ on $A(\mathcal{F}_1)$, then $\mu_2$ is called a measure extension of $\mu_1$ from $A(\mathcal{F}_1)$ to $A(\mathcal{F}_2)$ (or, less precisely, from $\mathcal{F}_1$ to $\mathcal{F}_2$); and a regular measure extension, if $\mu_2 \in M_R(\mathcal{F}_2)$.

If $\mu_1 \leq \mu_2 |$ on $\mathcal{F}_1$ and $\mu_1(X) = \mu_2(X)$, then $\mu_2$ is called a measure enlargement of $\mu_1$ from $A(\mathcal{F}_1)$ to $A(\mathcal{F}_2)$ (or, less precisely, from $\mathcal{F}_1$ to $\mathcal{F}_2$); and a regular measure enlargement, if $\mu_2 \in M_R(\mathcal{F}_2)$.

**Definition 2.5**

A real-valued function $\mu_\circ : \phi(X) \rightarrow [0, \infty)$, is called a finitely superadditive inner measure, if

1. $\mu_\circ(\emptyset) = 0$
2. [nondecreasing] $\forall A \subset B \subset X \Rightarrow \mu_\circ(A) \leq \mu_\circ(B)$, that is, $\mu_\circ \uparrow$
3. [finite superadditivity] $\forall A, B \subset X, A \cap B = \emptyset \Rightarrow \mu_\circ(A \cup B) \geq \mu_\circ(A) + \mu_\circ(B)$

A real-valued function $\mu^\circ : \phi(X) \rightarrow [0, \infty)$, is called a finitely subadditive outer measure, if it satisfies (1), (2) and

3'. [finite subadditivity] $\forall A, B \subset X, A \cap B = \emptyset \Rightarrow \mu^\circ(A \cup B) \leq \mu^\circ(A) + \mu^\circ(B)$

Let $\mu^\circ$ be a finitely subadditive outer measure on $(X, \mathcal{F})$. A set $E \subset X$ is said to be $\mu^\circ$-measurable, if

$$\mu^\circ(T) = \mu^\circ(T \cap E) + \mu^\circ(T \cap E^c), \quad \forall T \subset X.$$
We have the following theorem characterizing a normal lattice as a special case of the coallocation property:

**Theorem 2.1**

\( \mathcal{L} \) is normal \( \iff \forall \mathcal{L} \in \mathcal{L} \) s.t. \( L \leq L_1 \cup L_2 \), where \( L_1, L_2 \in \mathcal{L} \)

\[ \exists \hat{L}_1, \hat{L}_2 \in \mathcal{L} \text{ s.t. } L_1 \leq \hat{L}_1, L_2 \leq \hat{L}_2, \hat{L}_1 \cup \hat{L}_2 \leq L \]

**Proof:**

Suppose \( L_1, L_2 \in \mathcal{L} \), and \( \exists L_1 \cap L_2 = \emptyset \). Then by assumption, \( \exists \hat{L}_1, \hat{L}_2 \in \mathcal{L} \) such that \( \hat{L}_1 \cup L_1 \), \( \hat{L}_2 \cup L_2 \) and \( \hat{L}_1 \cup \hat{L}_2 = \emptyset \).

Thus, \( \hat{L}_1 \cup \hat{L}_2 = \emptyset \), we have \( \hat{L}_1 \cap \hat{L}_2 = \emptyset \). \( \therefore \) is normal.

The following results are obvious:

(1) \( \mathcal{L} \) is normal \( \iff \mathcal{L} \) coallocates itself \( \iff \mathcal{L} \) coseparates itself.

(2) \( \mathcal{L}_1 \) separates \( \mathcal{L}_2 \) \( \iff \mathcal{L}_1 \) semi-separates \( \mathcal{L}_2 \).

Furthermore, we have the following measure theoretic characterization of a normal lattice:

**Theorem 2.2**

\( \mathcal{L} \) is normal iff

\[ \forall \mu \in I(\mathcal{L}), \text{ s.t. on } \mathcal{L}, \mu \leq \nu_1 \in I_R(\mathcal{L}), \mu \leq \nu_2 \in I_R(\mathcal{L}) \implies \nu_1 = \nu_2. \]

**Theorem 2.3**

Suppose \( \mathcal{L}_1 \cup \mathcal{L}_2 \). Then \( \mathcal{L}_1 \) coseparates \( \mathcal{L}_2 \) \( \implies \mathcal{L}_2 \) coallocates \( \mathcal{L}_1 \).

**Proof:**

Suppose \( L_1 \cup \hat{L}_1, L_2 \cup \hat{L}_2 \in \mathcal{L} \). Then,

\( \hat{L}_1 \cup \hat{L}_2, L_1 \cap \hat{L}_2 \in \mathcal{L} \).

Now, \( (L_1 \cap \hat{L}_2) \cup (L_2 \cap \hat{L}_2) = L_1 \cap (L_2 \cap \hat{L}_2) = L_1 \cap (L_2 \cap \hat{L}_2) = \emptyset. \)

\( \mathcal{L}_1 \) coseparates \( \mathcal{L}_2 \) \( \iff \exists \hat{L}_1, \hat{L}_2 \in \mathcal{L} \) s.t. \( \hat{L}_1 \cup \hat{L}_2 = \emptyset \), and
L_1 \cap L_2 \subset L'_1 \text{ and } L_1 \cap L'_2 \subset L'_1.

Define \( L'_1 = L_1 \cap L'_1 \) and \( L'_2 = L_1 \cap L'_2 \), hence \( L'_1, L'_2 \in \mathcal{L}_1 \). And

\[ L'_1 \cup L'_2 = (L_1 \cap L'_1) \cup (L_1 \cap L'_2) = L_1 \cap (L'_1 \cup L'_2) = L_1 \cap (L'_1 \cap L'_2)' = L_1. \]

.: \( L_1 = L'_1 \cup L'_2 \).

Now

\[ L'_1 = L_1 \cap L'_1 \subset L_1 \cap (L_1 \cap L'_2)' = L_1 \cap (L_1 \cap L'_2) = (L_1 \cap L'_1) \cup (L_1 \cap L'_2) = L_1 \cap L'_1 \subset L'_1. \]

Thus, \( L'_1 \subset L'_1 \). Similarly, \( L'_1 \subset L'_1 \). Hence \( \mathcal{L}_1 \) coallocates \( \mathcal{L}_1 \).

**Theorem 2.4**

\[ \mathcal{L} \text{ countably paracompact } \implies M_\sigma(\mathcal{L}') \subset M_\sigma(\mathcal{L}). \]

**Proof:**

Suppose \( \forall n, \ L_n \in \mathcal{L}, \ L_n \downarrow \emptyset \)

\( \mathcal{L} \text{ c.p. } \implies \exists L_n \in \mathcal{L}, \text{ s.t. } L_n \subset L'_n, \ L'_n \downarrow \emptyset \)

\( \mu \in M_\sigma(\mathcal{L}') \iff \mu(L'_n) \rightarrow 0 \text{ as } L'_n \downarrow \emptyset \)

\[ \mu(L_n) \leq \mu(L'_n) \rightarrow 0, \implies \mu(L_n) \rightarrow 0 \]

Now \( L_n \downarrow \emptyset, \mu(L_n) \rightarrow 0 \implies \mu \in M_\sigma(\mathcal{L}). \text{ Hence, } M_\sigma(\mathcal{L}') \subset M_\sigma(\mathcal{L}). \]

**3. Measures on Coallocation Lattices**

In this section we extend some of the work of [8] and [2] on the unique extendability of a measure \( \mu \in M_\sigma(\mathcal{L}_1) \) to a measure \( \nu \in M_\sigma(\mathcal{L}_2) \) where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are lattices of subsets of \( X \). We note that it is not always necessary to assume that \( \mathcal{L}_1 \subset \mathcal{L}_2 \) nor that \( X \) belongs to the lattices in order for the main results of the coallocation theorem to hold (see Theorem 3.1). We first define two functions which form an inner-outer measure pair.

**Definition 3.1**

Suppose \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are lattices of subsets of \( X \) and \( \mu \in M(\mathcal{L}_1) \).

For all \( E \subseteq X \), define

\[ \mu^-(E) = \inf \{ \mu(L) : E \subseteq L, \ L \in \mathcal{L}_1 \} \]

and

\[ \mu^+(E) = \sup \{ \mu(L) : E \subseteq L, \ L \in \mathcal{L}_2 \} \]

We have the following:

**Theorem 3.1** [Coallocation theorem]

Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be lattices of subsets of \( X \neq \emptyset \). Suppose \( \mu \in M(\mathcal{L}_1) \). We have
(1) $\mu$ is a finitely superadditive inner measure

(2) $\mathcal{L}_2$ coallocates $\mathcal{L}_1 \Rightarrow \mu$ is finitely additive on $\mathcal{L}_1'$

(3) $\mathcal{L}_2$ coallocates $\mathcal{L}_1 \Rightarrow \mu^*$ is a finitely subadditive outer measure

(4) $\mu^* = \mu^*$ on $\mathcal{L}_1'$

In particular, if $X \in \mathcal{L}_1$ and $\emptyset \in \mathcal{L}_2$, then $\mu^*(X) = \mu_*(X) = \mu(X)

(5) $\mu \leq \mu^*$ on $\mathcal{L}_1$

(b) $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and $\mu \in \mathcal{M}_R(\mathcal{L}_1) \Rightarrow \mu_* = \mu$ on $\mathcal{L}_1'$; $\mu^* = \mu$ on $\mathcal{L}_1$

(6) Suppose $\mathcal{L}_2$ coallocates $\mathcal{L}_1$. $E \subseteq X$ is $\mu^*$-measurable

$\Rightarrow \forall L_2 \in \mathcal{L}_2, \mu^*(L_2) \geq \mu^*(L_2 \cap E) + \mu^*(L_2 \cap E^\complement)$

(7) Suppose $\mathcal{L}_2$ coallocates $\mathcal{L}_1$.

If either [a] $\mathcal{L}_1 \subseteq \mathcal{L}_2$

or [b] $\mathcal{L}_2$ semi-separates $\mathcal{L}_1$

then

1. every element of $\mathcal{L}_2^\complement$ is $\mu^*$-measurable

2. $\mu^*|_{\mathcal{L}_2}$ is a finitely additive measure on $A(\mathcal{L}_2)$

3. $\mu^*$ is $\mathcal{L}_1$-regular on $\mathcal{L}_2^\complement$

4. $\mu^* \in \mathcal{M}_R(\mathcal{L}_2)$.

Proof:

(1) The proof is standard and is therefore omitted.

(2) Let $A_2, B_2 \in \mathcal{L}_2$, and $L_1 \in \mathcal{L}$ s.t. $L_1 \subseteq A_2 \cup B_2 \in \mathcal{L}_2'$

$\mathcal{L}_1$ coallocates $\mathcal{L}_2 \Rightarrow$

$\exists A_1, B_1 \in \mathcal{L}_1$ s.t. $A_1 \subseteq A_2', B_1 \subseteq B_2'$, and $L_1 = A_1 \cup B_1$

$\therefore \mu(L_1) = \mu(A_1 \cup B_1)$

$\leq \mu(A_1) + \mu(B_1)$

$\leq \sup \{ \mu(A_1) : A_1 \supseteq A_2' \} + \sup \{ \mu(B_1) : B_1 \supseteq B_2' \}$

$= \mu_*(A_2') + \mu_*(B_2')$

Taking sup on the left hand side,

$\sup \{ \mu(L_1) : L_1 \subseteq A_2' \cup B_2' \} \leq \mu_*(A_2') + \mu_*(B_2')$

$\therefore \mu_*(A_2' \cup B_2') \leq \mu_*(A_2') + \mu_*(B_2') \Rightarrow$

$\mu_*$ is finitely subadditive on $\mathcal{L}_2^\complement$. Together with (1), $\mu_*$ is finitely additive on $\mathcal{L}_2^\complement$.

(3) The proof is also standard and is omitted.

(4) Let $L_2 \in \mathcal{L}_2'$.

$\mu^*(L_2) = \inf \{ \mu_*(\tilde{L}_2) : L_2 \subseteq \tilde{L}_2, \tilde{L}_2 \in \mathcal{L}_2 \}$

$\Rightarrow \mu^*(L_2) \leq \mu_*(L_2)$ ..............................................[i]
Now if $A_1^d \in \mathcal{L}_1^d$ and $L_1^d \subset A_1^d$, then by monotonicity of $\mu_*$,

$$\mu_*(L_1^d) \leq \mu_*(A_1^d)$$

$\Rightarrow \mu_*(L_1^d) \leq \inf \{ \mu_*(A_1^d) : L_1^d \subset A_1^d, A_2^d \in \mathcal{L}_2^d \} = \mu^-(L_1^d)$ \hspace{1cm} \text{[iii]}

[i] and [ii] $\Rightarrow \mu^- = \mu_*$ on $\mathcal{L}_2^d$.

If $X \in \mathcal{L}_1^d$, take $L_1 = X$, $\mu_*(X) = \sup \{ \mu(L_1) : X \subset L_1 \in \mathcal{L}_1^d \} = \mu(X)$.

If $\emptyset \in \mathcal{L}_2^d$, $X = \emptyset \in \mathcal{L}_2^d$ and $\mu^- = \mu_*$ on $\mathcal{L}_2^d \Rightarrow \mu^-(X) = \mu_*(X)$

Consequently, \( \mu^-(X) = \mu_*(X) = \mu(X) \).

(5)[a] Let $L_1 \in \mathcal{L}_1^d$ and $A_2 \in \mathcal{L}_2^d$, s.t. $L_1 \subset A_1^d$

\( \mu_*(A_1^d) = \sup \{ \mu(L_1) : A_1^d \supset L_1 \in \mathcal{L}_1^d \} \geq \mu(L_1) \)

Taking inf, \( \inf \{ \mu_*(A_1^d) : L_1 \subset A_1^d, A_2 \in \mathcal{L}_2^d \} \geq \mu(L_1) \)

i.e. \( \mu^-(L_1) \geq \mu(L_1) \). Or, \( \mu \leq \mu^- \) on $\mathcal{L}_1^d$.

(5)[b] Suppose $\mathcal{L}_1 \subset \mathcal{L}_2^d$, then $\mu^- = \mu_*$ on $\mathcal{L}_2^d \Rightarrow \mu^- = \mu_*$ on $\mathcal{L}_2^d$

.: if $\bar{L}_1 \in \mathcal{L}_1$, then $\mu^-(\bar{L}_1) = \mu_*(\bar{L}_1)$

Suppose $L_1 \subset A_1^d \subset A_1^d$, where $A_1 \in \mathcal{L}_1^d$, $A_2 \in \mathcal{L}_2^d$, $A_1 \subset A_2$

\( \mu_*(A_1^d) = \sup \{ \mu(\bar{L}_1) : A_1^d \subset \bar{L}_1 \in \mathcal{L}_1^d \} \)

But $\mu \in M_\mathcal{R}(\mathcal{L}_1) \Rightarrow \mu(A_1^d) = \sup \{ \mu(\bar{L}_1) : \bar{L}_1 \in \mathcal{L}_1^d \}$ whenever $A_1^d \subset \bar{L}_1$

\( \mu_*(A_1^d) = \mu(A_1^d) \) \( \forall A_1^d \in \mathcal{L}_1^d \), hence $\mu_* = \mu$ on $\mathcal{L}_1^d$.

Now, $\mu^-(L_1) = \inf \{ \mu_*(A_1^d) : L_1 \subset A_1^d, A_2 \in \mathcal{L}_2^d \}$

\[ \leq \inf \{ \mu_*(A_1^d) : L_1 \subset A_1^d, A_2 \in \mathcal{L}_2^d \} \quad (\because A_1^d \subset A_1^d) \]

\[ = \inf \{ \mu(A_1^d) : L_1 \subset A_1^d, A_2 \in \mathcal{L}_2^d \} \quad (\because \mu_* = \mu \text{ on } \mathcal{L}_2^d) \]

\[ = \mu(L_1) \quad (\because \mu \in M_\mathcal{R}(\mathcal{L}_1)) \]

$\therefore \mu^- \leq \mu_*$ on $\mathcal{L}_1$, and (5)[a] $\Rightarrow \mu \leq \mu^- \text{ on } \mathcal{L}_1$, $\therefore \mu^- = \mu$ on $\mathcal{L}_1$.

(6) "\text{fl}" Suppose $\forall A_2 \in \mathcal{L}_2^d$, we have $\forall E \subset X$,

$$\mu^- (A_2^d) \geq \mu^- (A_2^d \cap E) + \mu^- (A_2^d \cap E')$$

Suppose $T \subset X$, s.t. $T \subset A_2^d$, $A_2 \in \mathcal{L}_2^d$

$\mu^-(T) = \inf \{ \mu_*(A_2^d) : T \subset A_2^d, A_2 \in \mathcal{L}_2^d \}$

Now (4) $\Rightarrow$

$$\mu_*(A_2^d) = \mu^-(A_2^d)$$

\[ \geq \mu^-(A_2^d \cap E) + \mu^-(A_2^d \cap E') \quad \text{(by assumption)} \]

\[ \geq \mu^-(T \cap E) + \mu^-(T \cap E') \quad (\because T \subset A_2^d, \mu^- \text{ f.s.}) \]

Taking inf,

$$\mu^-(T) \geq \mu^-(T \cap E) + \mu^-(T \cap E')$$ \hspace{1cm} \text{[iii]}

$\mu^-$ is finitely subadditive $\Rightarrow$

$$\mu^-(T) = \mu^-(T \cap (E \cup E')) \leq \mu^-(T \cap E) + \mu^-(T \cap E')$$ \hspace{1cm} \text{[iv]}
[iii] and [iv] \[ \mu^-(T) = \mu^-(T \cap E) + \mu^-(T \cap E') \quad \forall T \subseteq X, \]
which is the definition of \( E \) to be \( \mu^- \)-measurable.

(6) "\( \Rightarrow \)" By the definition of \( E \) to be \( \mu^- \)-measurable, we have

\[ \mu^-(T) = \mu^-(T \cap E) + \mu^-(T \cap E') \quad \forall T \subseteq X, \]

But \( \mu^- \) is finitely subadditive, the above is equivalent to

\[ \mu^-(T) \geq \mu^-(T \cap E) + \mu^-(T \cap E') \quad \forall T \subseteq X, \]

In particular, take \( T = L_2 \subseteq L_2' \), we have

\[ \mu^-(L_2') \geq \mu^-(L_2 \cap E) + \mu^-(L_2 \cap E') \quad \forall L_2 \subseteq L_2'. \]

(7) Suppose \( L_2 \) coallocates \( L_1 \).

Let \( L_2' \subseteq L_2' \). To prove that \( L_2' \) is \( \mu^- \)-measurable, we have to show, by (6), that

\[ \mu^-(A_2') \geq \mu^-(A_2' \cap L_2') + \mu^-(A_2' \cap L_2) \quad \forall A_2 \subseteq L_2. \]

Let \( A_2, B_2 \subseteq L_2 \), s.t.

\[ P \subseteq A_2 \cap L_2 \quad \text{and} \quad Q \subseteq A_2 \cap P' \]

Thus, \( P \cup Q \subseteq (A_2 \cap L_2') \cup (A_2 \cap P') \subseteq A_2' \)

and \( P \cap Q \subseteq P \cap (A_2 \cap P') = \emptyset \)

\[ \mu^-(A_2') = \mu_*(A_2') \quad (\mu^- = \mu_*, \text{ on } L_2') \]

\[ \sup \left( \mu(P \cup Q) : A_2 \cap P \cup Q \subseteq L_2' \right) \]

\[ \geq \mu(P) + \mu(Q) \quad (P \cap Q = \emptyset) \]

\[ \mu^-(A_2') \geq \mu(P) + \mu_*(A_2' \cap P') \]

\[ \mu^-(A_2') \geq \mu_*(P) + \mu_*(A_2' \cap P') \]

(7)[a]: Suppose \( L_1 \subseteq L_2 \).

If \( L_1 \subseteq L_2 \), then \( P \subseteq L_1 \Rightarrow P \subseteq L_2 \quad \Rightarrow A_2 \cup P \subseteq L_2 \)

\[ A_2 \cap P' = (A_2 \cup P)' \subseteq L_2', \quad \mu^- = \mu_*, \text{ on } L_2', \quad :[v] \Rightarrow \]

\[ \mu^-(A_2') \geq \mu(P) + \mu^-(A_2' \cap P') \]

\[ \geq \mu(P) + \mu^-(A_2' \cap L_2) \quad (P \subseteq L_2') \]

\[ \mu^-(A_2') \geq \sup \left( \mu(P) : A_2 \cap L_2' \subseteq P \subseteq L_2 \right) + \mu^-(A_2' \cap L_2) \]

\[ = \mu_*(A_2' \cap L_2') + \mu^-(A_2' \cap L_2) \]

\[ = \mu^-(A_2' \cap L_2') + \mu^-(A_2' \cap L_2) \quad \forall A_2 \subseteq L_2 \quad (\mu^- = \mu_*, \text{ on } L_2') \]

We conclude, from (6), that every element of \( L_2' \) is \( \mu^- \)-measurable.

\[ \therefore A(L_2) = A(L_2') \subseteq \{ \mu^- \text{-measurable sets} \}. \]

By a standard Carathéodory
argument, \( \mu^*|_{\mathcal{L}_2} \) is a finitely additive measure on \( A(\mathcal{L}_2) \).

Suppose \( L_2 \in \mathcal{L}_2 \),

\[
\mu^*(L_2) = \mu_*(L_2) \quad \text{(by (4))}
\]

\[
= \sup \{ \mu(L_1) : L_1 \subseteq L_2, L_1 \in \mathcal{L}_1 \}
\]

\[
\leq \sup \{ \mu^*(L_1) : L_1 \subseteq L_2, L_1 \in \mathcal{L}_1 \} \quad \text{(by (5)[a])}
\]

But \( L_1 \subseteq L_2 \Rightarrow \mu^*(L_1) \leq \mu^*(L_2) \), taking sup \( \Rightarrow \)

\[
\sup \{ \mu^*(L_1) : L_1 \subseteq L_2, L_1 \in \mathcal{L}_1 \} \leq \mu^*(L_2)
\]

Hence,

\[
\mu^*(L_2) = \sup \{ \mu^*(L_1) : L_1 \subseteq L_2, L_1 \in \mathcal{L}_1 \}
\]

which means that \( \mu^* \) is \( \mathcal{L}_2 \)-regular on \( \mathcal{L}_2 \). Since \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \), \( \mu^* \) is also \( \mathcal{L}_2 \)-regular on \( \mathcal{L}_2 \). Now any element of \( A(\mathcal{L}_1) \) is, of the form

\[
\bigcup_{i=1}^{n} (A_i \cap B_i') \quad A_i, B_i \in \mathcal{L}_2
\]

Consequently, \( \mu^* \in M_R(\mathcal{L}_2) \).

(7)[b]: Suppose \( \mathcal{L}_2 \) semi-separates \( \mathcal{L}_1 \).

Now \( P \subseteq A_1 \cap L_1 \Rightarrow P \subseteq L_1 \Rightarrow P \cap L_2 = \emptyset \)

and \( P \in \mathcal{L}_1, L_2 \in \mathcal{L}_2 \).

\( \mathcal{L}_2 \) semi-separates \( \mathcal{L}_1 \)

\[
\exists L_2 \in \mathcal{L}_2 \text{ s.t. } P \subseteq L_2 \subseteq L_1 \quad \therefore P' \supset \tilde{L}_2' \supset L_2 \Rightarrow A_1 \cap P' \supset A_1 \cap \tilde{L}_2'
\]

From (v),

\[
\mu^*(A_1') \geq \mu(P) + \mu_*(A_1' \cap P')
\]

\[
\geq \mu(P) + \mu_*(A_1' \cap \tilde{L}_2')
\]

\[
= \mu(P) + \mu^*(A_1' \cap \tilde{L}_2') \quad \text{(by (4))}
\]

\[
\geq \mu(P) + \mu^*(A_1' \cap L_2)
\]

\[
\Rightarrow \mu^*(A_1') \geq \sup \{ \mu(P) : A_1' \cap L_2 \supset P \in \mathcal{L}_1 \} + \mu^*(A_1' \cap L_2)
\]

\[
= \mu_*(A_1' \cap L_1') + \mu^*(A_1' \cap L_2)
\]

\[
= \mu^*(A_1' \cap L_1') + \mu^*(A_1' \cap L_2), \quad \forall A_2 \in \mathcal{L}_2 \quad (\mu^* = \mu_* \text{ on } \mathcal{L}_2)
\]

We conclude, from (6), that every element of \( \mathcal{L}_2 \) is \( \mu^* \)-measurable.

\( : A(\mathcal{L}_2) = A(\mathcal{L}_2') \subseteq \{ \mu^* \text{-measurable sets} \} \). By a standard Carathéodory argument, \( \mu^*|_{\mathcal{L}_2} \) is a finitely additive measure on \( A(\mathcal{L}_1) \).

Let \( L_2 \in \mathcal{L}_2 \). Suppose \( L_1 \subseteq L_2 \subseteq L_1' \subseteq L_2 \). \( \mathcal{L}_2 \) semi-separates \( \mathcal{L}_1 \)

\[
\exists L_2 \in \mathcal{L}_2 \text{ s.t. } L_1 \subseteq L_2 \subseteq L_1' \quad \text{and } \mu^* = \mu \text{ on } \mathcal{L}_1,
\]

\[
\mu(L_1) = \mu^*(L_1)
\]

\[
\leq \mu^*(L_2) \leq \mu^*(L_1')
\]

\[
\therefore \mu(L_1) \leq \mu^*(L_2) \leq \mu^*(L_1')
\]

Taking sup,

\[
\sup(\mu(L_1) : L_1 \subseteq L_2, L_1 \in \mathcal{L}_1) \leq \sup(\mu^*(L_2) : L_2 \subseteq L_1', L_2 \in \mathcal{L}_2) \leq \mu^*(L_2)
\]
But \( \mu^*(L_2^1) = \sup(\mu(L_1) : L_1 \subseteq L_2^1, L_1 \in \mathcal{L}_1) \) Hence,
\[
\mu^*(L_2^1) = \sup(\mu(L_1) : L_1 \subseteq L_2^1, L_1 \in \mathcal{L}_1) = \sup(\mu^*(L_2^1) : L_2^1 \subseteq L_2^1, L_2^1 \in \mathcal{L}_2^1).
\]
\[\therefore \mu^* \text{ is } \mathcal{L}_1\text{-regular on } \mathcal{L}_1^1 \text{ and } \mathcal{L}_2\text{-regular on } \mathcal{L}_2^1, \text{ and consequently,} \]
\[
\mu^* \in M_R(\mathcal{L}_2).
\]

Note: If \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \), then \( \mathcal{L}_2 \) trivially semi-separates \( \mathcal{L}_1 \), (7)[a] \( \Rightarrow \) (7)[b].

**Corollary 3.1**

Suppose \( \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}, X \in \mathcal{L} \), and \( \mathcal{L} \) coallocates itself, \( (\mathcal{L} \text{ is normal}). \)
then
1. \( \mu^* \) is finitely additive and \( \mu^*(X) = \mu_*(X) = \mu(X) \)
2. \( \mu^*(L) + \mu_*(L') = \mu(X) \quad \forall L \in \mathcal{L} \)

Proof:
1. Direct consequences of Theorem 3.1.
2. From Theorem 3.1(4), \( \mu^* = \mu_*. \) on \( \mathcal{L}_2^1 = \mathcal{L}_1. \) \( \therefore \mu^*(L') = \mu_*(L') \quad \forall L \in \mathcal{L} \)

Now \( \mu^* \) is finitely additive,
\[
\mu^*(X) = \mu^*(L \cup L') = \mu^*(L) + \mu^*(L') = \mu^*(L) + \mu_*(L')
\]
But \( \mu^*(X) = \mu_*(X) = \mu(X), \therefore \mu^*(L) + \mu_*(L') = \mu(X). \)

The coallocation theorem leads to the following direct consequences whose proofs are omitted.

**Theorem 3.2** [Regular measure extension on coallocation lattices]

Suppose \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) and \( \mu \in M_R(\mathcal{L}_1). \) If \( \mathcal{L}_2 \) coallocates \( \mathcal{L}_1 \), then there exists a unique \( \nu \in M_R(\mathcal{L}_2) \), s.t. on \( \mathcal{L}_1 \), \( \mu = \nu|_{\mathcal{L}_1} \in M_R(\mathcal{L}_1). \)

Furthermore, \( \nu \) is \( \mathcal{L}_1\)-regular on \( \mathcal{L}_1^1 \). Note that \( \nu = \mu^*|_{\mathcal{L}_2}. \)

**Theorem 3.3** [Regular measure enlargement on coallocation lattices]

Suppose \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) and \( \mu \in M(\mathcal{L}_1). \) If \( \mathcal{L}_2 \) coallocates \( \mathcal{L}_1 \), then \( \exists \nu \in M_R(\mathcal{L}_2), \) s.t. \( \mu \leq \nu \) on \( \mathcal{L}_1 \) and \( \mu(X) = \nu(X). \)

**Theorem 3.4** [Regular measure enlargement on a normal lattice]

Suppose \( \mathcal{L} \) is normal and \( \mu \in M(\mathcal{L}). \) Then there exists a unique \( \nu \in M_R(\mathcal{L}), \) s.t. \( \mu \leq \nu \) on \( \mathcal{L} \) and \( \mu(X) = \nu(X). \)

Furthermore, if we impose a \( \sigma \)-smoothness condition on \( \mu \), we obtain the following:

**Theorem 3.5**

Suppose \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \), and \( \mathcal{L}_2 \) coallocates \( \mathcal{L}_1 \), and \( \mu \in M_R(\mathcal{L}_1), \).
\( \nu \in \mathcal{M}_R(L_2), \) where \( \nu \) is the regular measure extension of \( \mu. \) Then
\[
\mu \in \mathcal{M}_R^2(L_1) \Rightarrow \mu \in \mathcal{M}_R(L_1) \}
\) hence by Theorem 3.2,
\( \nu = \mu^* \mid L_2 = \mathcal{M}_R(L_2) \) is the unique regular measure extension of \( \mu. \)

Theorem 3.1(7) \( \Rightarrow \) is \( \mathcal{L}_1 \)-regular on \( L_1 \).

\[
\forall B_n^i \in L_2, B_n^i \upharpoonright \emptyset, \exists A_n \subseteq L_1, A_n \subseteq B_n^i, \ \forall \varepsilon > 0,
\]
\( \mu^*(B_n^i) \leq \mu^*(A_n) + \varepsilon/2 \)

\( \mu^* = \mu \) on \( L_1 \), since \( \mu \) is \( \mathcal{L}_1 \)-regular [Theorem 3.1(5)(b)]

\( \mu^*(B_n^i) \leq \mu^*(A_n) + \varepsilon/2 \) \[\] \[\]

Since \( A_n \subseteq B_n^i \upharpoonright \emptyset \), without the loss of generality, we may assume

\( A_n \downarrow \emptyset. \) Thus, \( \mu(A_n) \rightarrow 0 \) \( \Rightarrow \) is \( \sigma \)-smooth on \( L_1 \).

Hence, \( \| \Rightarrow \mu^*(B_n^i) \leq \varepsilon \)

Consequently,
\( \mu^*(B_n^i) \rightarrow 0 \ \forall B_n^i \in L_2 \)

Or,
\( \nu = \mu^* \mid L_2 \in \mathcal{M}_R(L_2) \cap \mathcal{M}_R(L_2) \).

In particular, if \( L_1 = L_2 = L \) is normal, we have

**Corollary 3.5**

Suppose \( L \) is normal, and \( \mu \in \mathcal{M}_\sigma(L), \ \nu \in \mathcal{M}_R(L), \ \nu \) is the regular measure enlargement of \( \mu, \ \mu \leq \nu \) on \( L \), \( \mu(X) = \nu(X). \) Then,
\( \nu \in \mathcal{M}_\sigma(L). \)

**Theorem 3.6**

Suppose \( L_1 \subseteq L_2 \), and \( L_2 \) coallocates \( L_1 \), and
\( L_2 \) is countably paracompact and normal. Suppose \( \mu \in \mathcal{M}_R(L_1), \) and
\( \nu \in \mathcal{M}_R(L_2), \) where \( \nu \) is the unique regular measure extension of \( \mu. \)

Then,
\( \nu \in \mathcal{M}_\sigma(L_2). \)

Proof:
\( L_2 \) c.p. \( \Rightarrow \forall B_n \subseteq L_2, B_n \upharpoonright \emptyset, \exists \tilde{B}_n \subseteq L_2, B_n \subseteq \tilde{B}_n \upharpoonright \emptyset \)

Theorem 3.5 \( \Rightarrow \nu \in \mathcal{M}_\sigma(L_2), \ \nu(\tilde{B}_n^i) \rightarrow 0 \Rightarrow \nu(B_n) \rightarrow 0 \ \forall B_n \subseteq L_2 \)

\( \Rightarrow \nu \) is \( \sigma \)-smooth on \( L_2 \), and since \( \nu \) is regular on \( L_2, \nu \in \mathcal{M}_R(L_2). \)

We now give two applications of the results on coallocation lattices to topological spaces.
1) **MEASURES ON A LOCALLY COMPACT HAUSDORFF SPACE**

Let $X$ be a **locally compact Hausdorff space** and $\mathcal{L}_1 = K_0$ is the collection of all compact $G_\delta$-sets, while $\mathcal{L}_2 = K$ is the collection of all compact sets. Note that in this case, $X$ does not belong to either $K_0$ or $K$, unless $X$ is compact. Then $K_0 \subset K$, and it can be shown that $K$ coallocates $K_0$. For any $\mu \in M_R(K_0)$, $\mu$ is $\sigma$-smooth, because $K_0$ is compact. Thus, $\mu \in M_R^0(K_0)$. By the coallocation theorem, we can extend $\mu$ uniquely to a regular measure $\nu$ which is also $\sigma$-smooth, because $K$ is compact. Hence, $\nu \in M_R^0(K)$.

2) **MEASURE ENLARGEMENT FROM ZERO SETS TO CLOSED SETS**

Suppose $X$ is a countably paracompact and normal topological space. Let $\mathcal{L}_1 = \mathcal{Z}$ (zero sets) and $\mathcal{L}_2 = \mathcal{B}$ (closed sets). That is, $\mathcal{B}$ is c.p. normal. $\mathcal{B} \subset \mathcal{Z}$ because all zero sets are closed $G_\delta$-sets, and disjoint closed sets can be separated by disjoint zero sets. Therefore, $\mathcal{B}$ is c.p. and normal. Thus, $\mathcal{B}$ coseparates $\mathcal{Z}$. Hence $\mathcal{Z}$ coallocates $\mathcal{B}$ [Theorem 2.3].

Let $\mu \in M_R(\mathcal{Z})$. Then by Theorem 3.2, there exists a unique regular measure extension $\nu \in M_R(\mathcal{B})$. Theorem 3.1(7)[a] implies $\nu$ on all open sets is $\mathcal{Z}$-regular.

Suppose $\mu \in M_R^0(\mathcal{Z})$. By Theorem 3.5, the unique regular measure extension is $\nu \in M_R(\mathcal{B}) \cap M_0(\mathcal{B})$. Now $\mathcal{B}$ is c.p., hence $\nu \in M_0(\mathcal{B})$ [Theorem 2.4]. Then, $\nu \in M_R^0(\mathcal{B})$. This is the result of Marik [5].

### 4. NORMAL LATTICES

In this section, we give further characterization of normal lattices and further consequences of a lattice being normal in terms of associated measures on the generalized algebra.

**Definition 4.1**

Let $\mathcal{L}$ be a lattice of subsets of $X$, and $\mu \in M(\mathcal{L})$. For $E \subset X$, define

$$\mu^*(E) = \inf \{ \mu(L') : E \subset \tilde{L}', \tilde{L} \in \mathcal{L} \}$$

$$\mu^*(E) = \inf \{ \sum_{n=1}^\infty \mu(L'_n) : E \subset \bigcup_{n=1}^\infty \tilde{L}_n, \tilde{L}_n \in \mathcal{L} \}$$

$$\mu^*(E) = \sup \{ \mu(L) : E \supset \tilde{L} \in \mathcal{L} \}$$

$$\mu^*(E) = \inf \{ \mu(L') : E \subset \tilde{L}', \tilde{L} \in \mathcal{L} \}$$

It is clear that if $\mu \in M_R(\mathcal{L})$, then $\mu = \mu^*$ on $A(\mathcal{L})$. 
THEOREM 4.1
Let $\mu, \nu \in M(\mathcal{L})$, such that $\mu(X) = \nu(X)$. Then
$$\mu \leq \nu \text{ on } \mathcal{L} \iff \mu \leq \nu \leq \nu' \leq \mu' \text{ on } \mathcal{L}$$

Proof:
It is obvious that $\mu \leq \nu \text{ on } \mathcal{L} \iff \nu' \leq \mu \text{ on } \mathcal{L}'$. Let $E \subset X$ s.t.
$E \subset \hat{L}'$, $\hat{L} \in \mathcal{L}$, $\nu(\hat{L}') \leq \mu(\hat{L}')$. Taking inf,
$$\inf(\nu(\hat{L}')) : \hat{L}' \in \mathcal{L}' \leq \inf(\mu(\hat{L}')) : \hat{L}' \in \mathcal{L}'$$
$\therefore \nu'(E) \leq \mu'(E)$. In particular, $E \in \mathcal{L} \Rightarrow \nu' \leq \mu' \text{ on } \mathcal{L}$. Hence,
$\mu \leq \nu \leq \nu' \leq \mu'$ on $\mathcal{L}$.

THEOREM 4.2
Suppose $\forall \mu \in I(\mathcal{L})$, $L_1, L_2 \in \mathcal{L}$,
$$\mu'(L_1) = 1 \text{ and } \mu'(L_2) = 1 \Rightarrow \mu'(L_1 \cap L_2) = 1$$
Then, $\mathcal{L}$ is normal.

Proof:
Suppose $\mathcal{L}$ is not normal. Then
$\exists \mu \in I(\mathcal{L})$, $\nu_1, \nu_2 \in I_R(\mathcal{L})$, s.t. $\mu \leq \nu_1 \text{ on } \mathcal{L}$, $\mu \leq \nu_2 \text{ on } \mathcal{L}$, but $\nu_1 \neq \nu_2$.
$\therefore \exists L_1, L_2 \in \mathcal{L}$, $L_1 \cap L_2 = \emptyset$,
$$\nu_1(L_1) = 1, \nu_2(L_1) = 0 \text{ and } \nu_1(L_2) = 0, \nu_2(L_2) = 1$$
Now if $L_1 \subset \hat{L}_1$, $\hat{L}_1 \in \mathcal{L}$, then $\nu_1(\hat{L}_1) = 1$. Since
$\mu \leq \nu_1 \text{ on } \mathcal{L} \iff \nu_1 \leq \mu \text{ on } \mathcal{L}'$, we have $\mu(\hat{L}_1) = 1 \Rightarrow \mu'(L_1) = 1$.
Similarly, if $L_2 \subset \hat{L}_2$, $\hat{L}_2 \in \mathcal{L}$, then $\mu'(L_2) = 1$.
Then, by assumption, $\mu'(L_1 \cap L_2) = 1$. But $L_1 \cap L_2 = \emptyset$, $\therefore \mu'(L_1 \cap L_2) = 0$
gives a contradiction. Consequently, $\mathcal{L}$ is normal.

THEOREM 4.3
Let $\nu \in M_R(\mathcal{L})$, $\rho \in M(\mathcal{L})$, s.t. $\nu(X) = \rho(X)$ and
on $\mathcal{L}'$, $\nu \leq \rho \in M_R(\mathcal{L}')$. Then
(1) $\rho \leq \nu = \nu' \leq \rho'$ on $\mathcal{L}$
(2) $\mathcal{L}$ is normal $\Rightarrow \nu = \nu' = \rho'$ on $\mathcal{L}$.

Proof:
(1) $\nu \leq \rho \text{ on } \mathcal{L}' \iff \rho \leq \nu \text{ on } \mathcal{L}$, and $\nu \in M_R(\mathcal{L}) \Rightarrow \nu = \nu'$. Hence
by Theorem 4.1, $\rho \leq \nu \leq \nu' \leq \rho'$ on $\mathcal{L}$.

(2) Suppose $\mathcal{L}$ is normal and $\exists L \in \mathcal{L}$ s.t. $\nu(L) < \rho'(L)$.
$\nu \in M_R(\mathcal{L}) \Rightarrow \forall \varepsilon > 0, \exists \tilde{L} \in \mathcal{L}$, $\tilde{L} \subset L'$, $\nu(\tilde{L}) + \varepsilon > \nu(L')$
$\therefore \nu(\tilde{L}') < \nu(L) + \varepsilon$ and $L \cap \tilde{L} = \emptyset$
By normality, $\exists L_a, L_b \in \mathcal{L}$, s.t. $L \subset L_a', \tilde{L} \subset L_b'$, $L_a' \cap L_b' = \emptyset$.
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\[ \nu(L) < \rho(L) \leq \rho(L_\alpha) \leq \rho(L_b) \leq \nu(L^\alpha) < \nu(L) + \varepsilon \]

\[ \Rightarrow \rho(L) \leq \nu(L) \text{ gives a contradiction.} \quad \therefore \nu = \nu' = \rho' \text{ on } \mathcal{L}. \]

**Theorem 4.4**

Let \( \mathcal{L} \) be a lattice of subsets of \( X \), and \( \mu \in \mathcal{M}_0(\mathcal{L}) \). Then,

1. \( \mu^n \leq \mu' \) everywhere
2. \( \mu' = \mu \) on \( \mathcal{L} \)
3. \( \mu \leq \mu'' \leq \mu' \) on \( \mathcal{L} \)
4. \( \mu(\bar{X}) = \mu''(\bar{X}) = \mu'(\bar{X}) \)
5. \( \mu(L^\alpha) + \mu'(L) = \mu(\bar{X}) \), \( \forall L \in \mathcal{L} \)
6. If \( \mathcal{L} \) is normal, then \( \mu \leq \mu'' \leq \mu' = \mu^\ast \) on \( \mathcal{L} \)
7. If \( \mathcal{L} \) is \( \delta \)-normal, then \( \mu'' = \mu' \) on \( \mathcal{L} \).

**Note:** The condition \( \mu \in \mathcal{M}_0(\mathcal{L}) \) is imposed, because when \( \mu \) is a 0-1 measure and if \( \mu \) is not \( \sigma \)-smooth, then \( \mu^n = 0 \).

**Proof:**

1. By definition of \( \mu^n \), the inf encompasses more sets than that of \( \mu' \), hence \( \mu^n \leq \mu' \) everywhere.
2. Take \( E = \bar{L} \in \mathcal{L} \), \( \therefore \mu' = \mu \) on \( \mathcal{L} \). In particular,
   \[ \mu'(\bar{X}) = \mu(\bar{X}) \] ........................ [i]
3. From (1) and [i], we have \( \mu^n(\bar{X}) \leq \mu(\bar{X}) \). We now show that \( \mu^n(\bar{X}) = \mu(\bar{X}) \). For suppose \( \bar{X} = \bigcup_{i=1}^{\infty} L_i \), pairwise disjoint \( L_i \in \mathcal{L} \), and \( \sum_{i=1}^{\infty} \mu(L_i) < \mu(\bar{X}) \), but \( \sum_{i=1}^{\infty} \mu(L_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(L_i) \geq \lim_{n \to \infty} \mu(\bigcup_{i=1}^{n} L_i) = \mu(\bar{X}) \).
4. Since \( \bigcup_{i=1}^{\infty} L_i \in \mathcal{L} \) and \( \bigcup_{i=1}^{\infty} L_i \uparrow_X \), or \( \bigcap_{i=1}^{\infty} L_i \downarrow \emptyset \), also \( \mu \in \mathcal{M}_0(\mathcal{L}) \).
5. Taking the inf of the above, we have \( \mu^n(\bar{X}) \geq \mu(\bar{X}) \).
6. Consequently, \( \mu^n(\bar{X}) = \mu(\bar{X}) \) ........................ [ii]
7. Now suppose \( \exists L \in \mathcal{L}, \mu(L) > \mu^n(L) \),
   \[ \mu^n(\bar{X}) = \mu^n(\bar{L} \cup L') \leq \mu^n(L) + \mu^n(L') \]
   \[ \leq \mu^n(L) + \mu(L') \quad (\because \mu^n \leq \mu \text{ on } \mathcal{L}) \]
   \[ < \mu(L) + \mu(L') \quad \text{(by assumption)} \]
   \[ = \mu(\bar{X}) \quad \text{contradicting [ii]} \]
   \[ \therefore \mu \leq \mu^n \text{ on } \mathcal{L}. \text{ Together with (1), we have } \mu \leq \mu^n \leq \mu' \text{ on } \mathcal{L}. \]
8. [i] and [ii] \( \Rightarrow \mu(\bar{X}) = \mu^n(\bar{X}) = \mu'(\bar{X}). \)
(5) \( \forall \varepsilon > 0, \mu_*(L') - \varepsilon < \mu(\hat{L}) \) where \( L' \supset \hat{L} \in \mathcal{L} \)

\[ \therefore \mu(\hat{X}) - \mu_*(L') > \mu(\hat{L}') - \varepsilon \]

Taking inf, we have

\[ \mu(\hat{X}) - \mu_*(L') \geq \mu'(L) \quad \text{............................. [iii]} \]

If \( L' \supset \hat{L} \), \( \mu_*(L') \geq \mu(L') \geq \mu(\hat{L}) \) since \( \mu_* \leq \mu \) on \( \mathcal{L}' \)

\[ \therefore \mu(\hat{X}) - \mu_*(L') \leq \mu(\hat{X}) - \mu(\hat{L}) = \mu(\hat{L}') \]

Taking inf, we have

\[ \mu(\hat{X}) - \mu_*(L') \leq \inf (\mu(\hat{L}) : L \subset \hat{L'}, \hat{L} \in \mathcal{L}) = \mu'(L) \quad \text{..... [iv]} \]

[iii] and [iv] \implies \( \mu(\hat{X}) - \mu_*(L') = \mu'(L) \quad \forall L \in \mathcal{L} \).

(6) \( L \) is normal, then Corollary 3.1(2) and (5) imply \( \mu^-(L) = \mu'(L) \quad \forall L \in \mathcal{L} \)

Now \( \mu^* = \mu^- \) on \( \mathcal{L} \), and from (3), \( \mu \leq \mu^* \leq \mu' = \mu^- \) on \( \mathcal{L} \).

(7) From (1), \( \mu^* \) is everywhere. Suppose \( \exists L \in \mathcal{L} \), s.t. \( \mu^*(L) < \mu'(L) \)

Then \( L \subset \bigcup_{n=1}^{\infty} L_n \), \( L_n \in \mathcal{L} \). \( \mathcal{L} \) is \( \delta = \bigcap_{n=1}^{\infty} L_n \in \mathcal{L} \).

Let \( D = \bigcap_{n=1}^{\infty} L_n \), \( L \cap D = \emptyset \). \( \mathcal{L} \) is normal \( \Rightarrow \exists A, B \in \mathcal{L} \), s.t.

\[ L \subset A', \quad D \subset B', \quad A' \cap B' = \emptyset \quad \therefore \quad L \subset A' \subset B \subset D' = \bigcup_{n=1}^{\infty} L_n \]

\[ \mu'(L) \leq \mu'(A') = \mu(A') \leq \mu(B) \leq \mu^*(B) \leq \sum_{n=1}^{\infty} \mu(L_n) \quad \text{from (3)} \]

Taking inf, and since \( L \subset \bigcup_{n=1}^{\infty} L_n \),

\[ \mu'(L) \leq \inf (\sum_{n=1}^{\infty} \mu(L_n) : L \subset \bigcup_{n=1}^{\infty} L_n, L_n \in \mathcal{L}) = \mu^*(L) \]

gives a contradiction. \( \therefore \mu^*(L) = \mu'(L) \), or \( \mu^* = \mu' \) on \( \mathcal{L} \).

**Theorem 4.5**

Let \( \mathcal{L} \) be a lattice of subsets of \( X \), and let \( \mu \in \mathcal{M}_0(\mathcal{L}) \), \( \rho \in \mathcal{M}(\mathcal{L}) \), s.t. \( \mu \leq \rho \) on \( \mathcal{L} \), \( \mu(\hat{X}) = \rho(\hat{X}) \).

If \( \mathcal{L} \) is countably paracompact and normal, then \( \rho \in \mathcal{M}_0(\mathcal{L}) \).

**Proof:**

Let \( L_n \upharpoonright \emptyset, \quad L_n \in \mathcal{L}, \quad \forall n \)

\( \mathcal{L} \) c.p. \( \Rightarrow \exists \hat{L}_n \in \mathcal{L}, \quad L_n \subset \hat{L}_n \upharpoonright \emptyset \quad \therefore L_n \cap \hat{L}_n = \emptyset \)

\( \mathcal{L} \) normal \( \Rightarrow \exists A_n, B_n \in \mathcal{L}, \quad L_n \subset A_n, \quad \hat{L}_n \subset B_n, \quad A_n \cap B_n = \emptyset \).

Or, \( L_n \subset A_n \subset B_n \subset \hat{L}_n \upharpoonright \emptyset, \quad \therefore \rho(L_n) \leq \rho(A_n) \leq \mu(A_n) \leq \mu(B_n) \rightarrow 0 \)

(one may assume, with the loss of generality, \( B_n \upharpoonright \emptyset \)).

\( \therefore \rho \leq \mu \) on \( \mathcal{L}' \); \( B_n \upharpoonright \emptyset \) and \( \mu \in \mathcal{M}_0(\mathcal{L}') \). \( \therefore \rho(L_n) \rightarrow 0 \), or \( \rho \in \mathcal{M}_0(\mathcal{L}) \).

**Theorem 4.6**

Suppose \( \mathcal{L}_1 \subset \mathcal{L}_2 \), and \( \mathcal{L}_1 \) separates \( \mathcal{L}_2 \). Then,

\( \mathcal{L}_1 \) normal \( \iff \mathcal{L}_2 \) normal.

**Proof:**

"\( \Rightarrow \)" Suppose \( \mathcal{L}_1 \) is normal.
Let \( \mu \in \mu(\mathcal{L}_2), \ \nu_a, \nu_b \in \mu(\mathcal{L}_2), \) s.t. \( \mu \leq \nu_a \) on \( \mathcal{L}_2, \ \mu \leq \nu_b \) on \( \mathcal{L}_2. \)

Then \( \mu \in \mu(\mathcal{L}_1), \ \nu_a, \nu_b \in \mu(\mathcal{L}_1), \) and \( \mu \leq \nu_a \) on \( \mathcal{L}_1, \ \mu \leq \nu_b \) on \( \mathcal{L}_1. \) \( \mathcal{L}_1 \) normal \( \iff \nu_a = \nu_b \) [Theorem 2.2].

Extend \( \nu_a \) and \( \nu_b \) to \( \mathcal{L}_2 \Rightarrow \nu_a = \nu_b \Rightarrow \mathcal{L}_2 \) normal,

\( A \) separates \( \mathcal{L}_1, \) the extension is unique.

Suppose \( \mathcal{L}_2 \) is normal.

Let \( \mu \in \mu(\mathcal{L}_1), \ \nu_a, \nu_b \in \mu(\mathcal{L}_1), \) s.t. \( \mu \leq \nu_a \) on \( \mathcal{L}_2, \ \mu \leq \nu_b \) on \( \mathcal{L}_2. \)

Extend \( \mu \) to \( \lambda \in \mu(\mathcal{L}_2), \) and \( \nu_a, \nu_b \) to \( r_a, r_b \in \mu(\mathcal{L}_2), \) respectively. We now show that \( \lambda \leq r_a \) on \( \mathcal{L}_2, \) and \( \lambda \leq r_b \) on \( \mathcal{L}_2. \)

For suppose \( \exists L_2 \in \mathcal{L}_2 \) s.t. \( \lambda(L_2) = 1 \) but \( r_a(L_2) = 0. \) Then \( r_a(L_2) = 1. \) But \( r_a \in \mu(\mathcal{L}_2), \ \exists \tilde{L}_2 \in \mathcal{L}_2, \) s.t. \( \tilde{L}_2 \subseteq L_2, \ r_a(L_2) = 1 \)

Since \( \mathcal{L}_1 \) separates \( \mathcal{L}_2 \Rightarrow \exists L_1 \in \mathcal{L}_1, \) s.t. \( L_1 \subseteq L_2, \mathcal{L}_1 \subseteq \tilde{L}_2 \)

\( \therefore 1 = \lambda(L_2) \leq \lambda(L_1) \leq \mu(L_1) \leq \nu_a(L_1) \leq r_a(L_1) \leq r_a(L_2) \)

Thus, \( r_a(L_2) = 1 \) or \( r_a(L_2) = 0 \) contradicting \( r_a(L_2) = 1 \)

\( \therefore \) \( \lambda \leq r_a \) on \( \mathcal{L}_2. \) Similarly, \( \lambda \leq r_b \) on \( \mathcal{L}_2. \) Since \( \mathcal{L}_2 \) is normal,

\( r_a = r_b \Rightarrow r_a = 0 \), i.e. \( \nu_a = \nu_b \Rightarrow \mathcal{L}_1 \) is normal.

**Theorem 4.7**

Suppose \( \mathcal{L}_1 \subseteq \mathcal{L}_2, \) and \( \mu \in \mu(\mathcal{L}_1), \ \nu \in \mu(\mathcal{L}_2), \) s.t.

\( \mu(X) = \nu(X), \ \nu \mid \mathcal{L}_1 = \mu. \)

Then \( \mathcal{L}_1 \) separates \( \mathcal{L}_2 \Rightarrow \nu \) is \( \mathcal{L}_1 \)-regular on \( \mathcal{L}_2. \)

**Proof:**

\( \nu \in \mu(\mathcal{L}_2), \ \forall L_2 \in \mathcal{L}_2, \ \nu(L_2) = \sup( \nu(L_2) : L_2 \supset \tilde{L}_2 \subseteq \tilde{L}_2 \in \mathcal{L}_2) \)

\( \forall \varepsilon > 0, \ L_2 \supset \tilde{L}_2 \subseteq \mathcal{L}_2, \ \nu(L_2) < \nu(\tilde{L}_2) < \varepsilon \)

\( \exists \tilde{L}_1 \subseteq \tilde{L}_2 \subseteq \mathcal{L}_1, \ L_2 \supset \tilde{L}_1 \subseteq \mathcal{L}_1, \ L_2 \cap \tilde{L}_1 = \emptyset \)

\( \nu(L_2) < \nu(\tilde{L}_2) + \varepsilon \)

\( \leq \nu(\tilde{L}_1) + \varepsilon \) (\( \tilde{L}_2 \supset \tilde{L}_1 \))

\( = \mu(\tilde{L}_1) + \varepsilon \) (\( \nu \mid \mathcal{L}_1 = \mu \))

Taking \( \sup, \ \nu(L_2) = \sup( \mu(\tilde{L}_1) : L_2 \supset \tilde{L}_1 \subseteq \mathcal{L}_1), \ \forall L_2 \in \mathcal{L}_2 \)

i.e. \( \nu \) is \( \mathcal{L}_1 \)-regular on \( \mathcal{L}_2. \)
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REFERENCES


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