AN INFINITE VERSION OF THE PÓLYA ENUMERATION THEOREM

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ABSTRACT. Using measure theory, the orbit counting form of Pólya's enumeration theorem is extended to countably infinite discrete groups.

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1. INTRODUCTION.

Let $G$ be countable discrete group acting as permutations on a countable set $D$. Let $S$ be a finite set with cardinality, $|S| = N$. Denote by $S^D$ the set of functions from $D$ to $S$. For $\gamma \in S^D$ define $g_\gamma \in S^D$ by $g_\gamma(d) = \gamma(g^{-1}d)$. For a subgroup $K$ of $G$ let $\Delta_K$ be a set of representatives for the orbits of $K$ in $S^D$. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{\mathbf{e}_\gamma: \gamma \in S^D\}$ and inner product $\langle \cdot, \cdot \rangle$. Define a unitary representation of $G$ on $\mathcal{H}$ by $\pi(g)e_\gamma = e_{\gamma g}$.

The number of orbits of $G$ in $S^D$ is denoted by $|\Delta_G|$. For finite $G$ and $D$ this can be counted by the Pólya enumeration theorem. Specifically, for each $g \in G$, let $c_i(g)$ be the number of cycles of length $i$ in the representation of $g$ as a product of disjoint cycles in $D$ and let $M(g) = \frac{1}{|G|} \sum_{g \in G} M(g)$. Denote by $\sigma P_G$ the value $\sigma P$ at $\gamma = N$, $i = 1$ to $n$. Pólya's enumeration theorem, see Pólya [1], says that $|\Delta_G| = \sigma P_G$.

Define the operator $T_G$ on $\mathcal{H}$ by $T_G = \frac{1}{|G|} \sum_{g \in G} \pi(g)$. Then it can also be shown, see Williamson [2], that $|\Delta_G| = \text{trace}(T_G \text{ on } \mathcal{H})$. It is these two ways of measuring a set of representatives for orbits that we extend to infinite $G$ and $D$.

2. THE MAIN RESULTS.

If we view $S$ as a finite group with the discrete topology, then $S^D$ is a compact group in the product topology. Let $\mu$ be normalized Haar measure on $S^D$.

For $g \in G$ and $\gamma \in S^D$ define $f(\gamma) = \langle \pi(g)e_\gamma, e_\gamma \rangle$. Then $f(\gamma) = 1$ if $g_\gamma = \gamma$ otherwise.

Lemma 1. $f$ is measurable.

Proof. Let $f_\gamma(d) = \begin{cases} 1 & \text{if } \gamma(g^{-1}d) = \gamma(d) \\ 0 & \text{otherwise} \end{cases}$ and $h_\gamma(\gamma) = \prod_{i=1}^{n} f_\gamma(d_i)$. Then $h_\gamma$ is measurable for all $n$. Now $g_\gamma = \gamma$ if and only if $\gamma$ is constant on the orbits of $g$. But this happens if and only if $\gamma(g^{-1}d) = \gamma(d)$ for all $d \in D$. Therefore $f(\gamma) = 1$ if and only if $f_\gamma(d_i) = 1$ for all $i$.
for all $i$. This shows that $f(\gamma) = \lim_{n \to \infty} b_n(\gamma)$ and therefore measurable by Hewett and Stromberg [3, 22.24b].

We write $D = \{d_1, d_2, d_3, \ldots\}$ and let $D_n = \{d_1, \ldots, d_n\}$. Let $< g >$ be the subgroup generated by $g$ and $< g > d$ the orbit of $d$ under $< g >$. For each $n$ and each $k \leq n$ let $c_k^n(g)$ be the number of distinct cycles of $g$ such that $|< g > d \cap D_n| = k$. Form the monomial $M^*(g) = \frac{1}{N} y_1^{i_1^{(n)}} y_2^{i_2^{(n)}} \cdots y_n^{i_n^{(n)}}$.

**Lemma 2.** \[ \int_{S^D} \phi(\gamma) d\mu(\gamma) = \lim_{n \to \infty} \mu M^*(g). \]

**Proof.** From the proof of Lemma 1 we saw that $\int_{S^D} \phi(\gamma) d\mu(\gamma) = \lim_{n \to \infty} h_n(\gamma)$. So by the dominated convergence theorem, $\int_{S^D} \phi(\gamma) d\mu(\gamma) = \lim_{n \to \infty} \int h_n(\gamma) d\mu(\gamma)$. But now $h_n(\gamma) = 1$ if and only if $\gamma$ is constant on the intersection of the orbits of $g$ with $D_n$ otherwise $h_n(\gamma) = 0$. Let $B_n = \{\gamma: \gamma \text{ is constant on the intersection of the orbits of } g \text{ with } D_n\}$. Then $\int h_n(\gamma) d\mu(\gamma) = \mu(B_n)$. Since there are $N$ choices for the value of $\gamma$ on each orbit meeting $D_n$ and no restrictions on $\gamma$ outside $D_n$, we get $\mu(B_n) = \frac{1}{N} N^{i_1^{(n)}} \cdots N^{i_n^{(n)}} = \sigma M^*(g)$. \hfill \Box

Let $G_\sigma$ be the subgroup of $G$ consisting of all those $g \in G$ having only a finite number of cycles in $D$ of length greater than 1.

**Lemma 3.** \[ \int_{S^D} \phi(\gamma) d\mu(\gamma) = 0 \text{ for all } g \notin G_\sigma. \]

**Proof.** Suppose $g \notin G_\sigma$. Then there either exists $k_\sigma$ such that $c_{k_\sigma}^*(g) \to \infty$ as $n \to \infty$ or there exists an increasing sequence $\{k_n\}$ such that $c_{k_n}^*(g) \geq 1$. In the first case, for $n \geq k_n$, $\sum_{i=1}^{n} c_i^*(g) = \sum_{i=1}^{n} (i-1) c_i^*(g) \leq c_{k_n}^*(g)$. So with $B_n$ as in the proof of Lemma 2, we get $0 \leq \int_{S^D} \phi(\gamma) d\mu(\gamma) = \lim_{n \to \infty} \mu(B_n) \leq \lim_{n \to \infty} N^{-(k_n-1)} = 0$. In the second case we get $n-\sum_{i=1}^{n} c_i^*(g) \leq k_n-1$ and so $0 \leq \int_{S^D} \phi(\gamma) d\mu(\gamma) = \lim_{n \to \infty} \mu(B_n) \leq \lim_{n \to \infty} N^{-(k_n-1)} = 0$. \hfill \Box

For each $k$ let $F_k = \{g \in G: g d_i = d_i \text{ for all } i > k\}$. Then $\{F_k\}$ is a nondecreasing sequence of subgroups with $\bigcap_{k=1}^{\infty} F_k = G_\sigma$. Suppose $G = \{g_1, g_2, \ldots\}$ and let $G_m = \{g_1, \ldots, g_m\}$. Assume $G$ is ordered in such a way that there exists a subsequence $\{m_k\}$ with $G_m \cap G_{m_k} = F_k$.

Let $F$ be a finite subset of $G$. Define the $n^{th}$ cycle index of $F$ to be the polynomial $P_F^* = \frac{1}{|F|} \sum_{g \in F} M^*(g)$. Define the operator $T_F$ on $\mathbb{K}$ by $T_F = \frac{1}{|F|} \sum_{g \in F} \phi(g)$. Write $P_m^*$ for $P_{0_m}^*$ and $T_m$ for $T_{G_{m}}$. 


THEOREM 4. \( \Delta_{G_0} \) is closed and
\[
\mu(\Delta_{G_0}) = \lim_{k \to \infty} \left\{ \frac{m_k}{|G_k \cap G_0|} \lim_{n \to \infty} \sigma P_{m_k} \right\} = \lim_{k \to \infty} \frac{m_k}{|G_k \cap G_0|} \int_{S^D} < T_{m_k} e_\gamma e_\gamma > d\mu(\gamma) .
\]

PROOF. Fix \( k \) and let \( D_k = \{ d_{k+i}, d_{k+i+1}, \ldots \} \). If \( \alpha_1, \ldots, \alpha_n \) are representatives for the orbits of \( F_k \) in \( S^D \), then \( \Delta_{F_k} = \{ (\alpha_1, \ldots, \alpha_n) \} \). Therefore \( \Delta_{F_k} \) is closed and \( \mu(\Delta_{F_k}) = \frac{8}{N^k} \). Let \( H_k \) be a Hilbert space with orthonormal basis \( \{ e_\alpha, \alpha \in S^D \} \). By Williamson [2],
\[
s = \text{trace}(T_{F_k} \text{ on } H_k) = \sigma P_{F_k},
\]
where \( P_{F_k} \) is the usual cycle index of \( F_k \) on \( D_k \). Note that \( \sigma P_{F_k} = N^k P_{F_k} \) for all \( n \geq k \). By Lemma 3, \( \frac{m_k}{|G_k \cap G_0|} \lim_{n \to \infty} \sigma P_{m_k} = \lim_{n \to \infty} \sigma P_{m_k} \). Therefore
\[
\frac{m_k}{|G_k \cap G_0|} \int_{S^D} < T_{m_k} e_\gamma e_\gamma > d\mu(\gamma) .
\]
Since \( F_k \subseteq G_k \) we can assume that \( \Delta_{G_k} \subseteq \Delta_{F_k} \) for all \( k \). Therefore \( \Delta_{G_k} = \bigcap_{k=1}^{\infty} \Delta_{F_k} \). We claim that \( \Delta_{G_k} = \bigcap_{k=1}^{\infty} \Delta_{F_k} \). To see this suppose that \( \gamma \in \Delta_{F_k} \) for all \( k \). Then there exists \( \gamma' \in \Delta_{G_k} \) and \( g \in G_k \) such that \( \gamma = g \gamma' \). Since \( G_k = \bigcup_{k=1}^{\infty} F_k \) there exists \( k \) such that \( g \in F_k \). Therefore \( \gamma \) and \( \gamma' \) represent the same orbit of \( F_k \) in \( S^D \). Since \( \gamma \) and \( \gamma' \in \Delta_{F_k} \) we get \( \gamma = \gamma' \). This proves the claim.

It follows that \( \Delta_{G_k} \) is closed and hence measurable. Therefore \( \mu(\Delta_{G_k}) = \lim_{k \to \infty} \mu(\Delta_{F_k}) \). This completes the proof of the theorem.

Suppose now that \( G \) is in no particular order. We show how to compute \( \mu(\Delta_{G_k}) \). Let \( A_m = G_m \cap G_0 \) and \( T_{A_m} = (T_{A_m})^n \).

THEOREM 5. \( \mu(\Delta_{G_k}) = \lim_{n \to \infty} \lim_{n \to \infty} \int_{S^D} < T_{A_m} e_\gamma e_\gamma > d\mu(\gamma) .
\]

PROOF. Exists \( m_0 \) so that \( 1 \in G_{m_0} \). Fix \( m \geq m_0 \) and let \( H_m \) be the subgroup of \( G_m \) generated by \( A_m \). Define a probability measure \( \nu \) on \( H_m \) by \( \nu(g) = \frac{1}{|A_m|} \) if \( g \in A_m \) and \( \nu(g) = 0 \) otherwise. Let \( \nu^n \) be the \( n \)-fold convolution of \( \nu \) with itself and \( U \) the uniform probability measure on \( H_m \). Then by Diaconis [4, pg23], \( \| \nu^n - U \| \to 0 \) where \( \| \cdot \| \) is the total variation norm. If we extend the representation \( \pi \), in the usual way, to the set of measures on \( H_m \) we get \( \pi(\nu^n) = (T_{A_m})^n = T_{A_m^n} \) and \( \pi(U) = T_{H_m} \). It follows, therefore, that
\[
\lim_{n \to \infty} < T_{A_m} e_\gamma e_\gamma > = < T_{H_m} e_\gamma e_\gamma > \text{ for all } \gamma \in S^D .
\]
By the dominated convergence theorem,
\[
\lim_{n \to \infty} \int_{S^D} < T_{A_m} e_\gamma e_\gamma > d\mu(\gamma) = \int_{S^D} < T_{H_m} e_\gamma e_\gamma > d\mu(\gamma) .
\]
Then as in the proof of Theorem 4, we get \( \mu(\Delta_{H_m}) = \int_{S^D} < T_{H_m} e_\gamma e_\gamma > d\mu(\gamma) . \) The result follows since \( G_m = \bigcup_{m=1}^{\infty} H_m . \)
3. EXAMPLE.

Suppose $D = \bigcup_{n=1}^{\infty} D_n$, where the $D_n$ are disjoint and finite and that $G$ sends $D_n$ into itself. Then if $G_n$ is $G$ restricted to $D_n$, $G$ is isomorphic to the product $\prod_{n=1}^{\infty} G_n$. In this case the product measure $\mu$ on $S^0$ need no longer come from uniform measures on $S$.

Let $S = \{s_1, \ldots, s_k\}$ and let the measure $\nu$ on $S$ be defined by $\nu(s_i) = a_i$. If $|D_n| = m_n$ define the measure $\mu_n$ on $S^D$ by $\mu_n = \prod_{i=1}^{m_n} \nu$. Let $\Delta_n$ be representatives for the orbits of $G_n$ in $S^D$ and $P_{G_n}$ the cycle index. Then using the pattern inventory from Pólya's enumeration theorem, see Pólya and Read [1], we get $\mu_n(\Delta_n) = P_{G_n} \left( \sum_{i=1}^{k} a_i, \sum_{i=1}^{k} a_i^2, \ldots, \sum_{i=1}^{k} a_i^m \right)$. Let $\mu = \prod_{n=1}^{\infty} \mu_n$ and let $\Delta$ be representatives for the orbits of $G$ in $R^0$. Then, as in the proof of Theorem 4, we get that $\mu(\Delta) = \lim_{n \to \infty} \prod_{k=1}^{n} \mu_n(\Delta_n)$. Note that when $a_i = 1/k$, $i = 1, \ldots, k$ and $|D_n| = n$ we get $\mu_n(\Delta_n) = \sigma P_{G_n}$, which is the situation in Theorem 4.

Now consider the plane tiled by one unit square tiles with sides parallel to the axis and center the coordinates $(m, n)$, $m$ and $n$ integers. We color the tiles black or white and compute the measure of the orbits of two groups of symmetries acting on the set of such tilings. For $m$ a positive integer let $D_m = \{\text{tiles with centers } (\pm m, k) \text{ or } (k, \pm m): k = m, m+1, \ldots, m-1, m\}$.

Let $G_n = \prod_{k=1}^{2n+1} \mathbb{Z}_2$ act on $D_n$ by interchanging tiles with central coordinates $(\pm n^2, k)$, $k = -n^2, \ldots, n^2$ and let $H_n = \prod_{k=1}^{2n+1} \mathbb{Z}_2$ act on $D_n$ by interchanging tiles with central coordinates $(\pm n^2, k)$, $k = -n, \ldots, n$. Now let $G = \prod_{n=1}^{\infty} G_n$ and $H = \prod_{n=1}^{\infty} H_n$. With $S = \{\text{black, white}\}$, we define probability measures $\mu_n$ on $S^D$ by $\mu_n = \prod_{n=1}^{m_n} \gamma_k$, where $\nu_k(\text{black}) = \sqrt[1/2]{\frac{\exp\left(-\frac{1}{n(2n+1)}\right)}{n^2}} + \frac{1}{2}$ and $\nu_k(\text{white}) = 1 - \nu_k(\text{black})$. Let $\Delta(G_n)$ and $\Delta(H_n)$ be representatives for the orbits of $G_n$ and $H_n$ respectively on $S^D$ and let $\Delta(G)$ and $\Delta(H)$ be representatives for the orbits of $G$ and $H$ respectively on $S^0$. Then $\mu_n(\Delta(H_n)) = \exp(-1/n^2)$ and so $\mu(\Delta(H)) = \lim_{n \to \infty} \prod_{n=1}^{m} \mu_n(\Delta(H_n)) > 0$. But $\mu_n(\Delta(G_n)) = \exp\left(-\frac{2n^2+1}{2n^2+n^2}\right)$ and so $\mu(\Delta(G)) = \lim_{n \to \infty} \prod_{n=1}^{m} \mu_n(\Delta(G_n)) = 0$.

REFERENCES


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