ON APPROXIMATION IN THE L_p -NORM BY HERMITE INTERPOLATION

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(Received September 7, 1990 and in revised form January 9, 1991)

ABSTRACT. L_p -approximation by the Hermite interpolation based on the zeros of the Tchebycheff polynomials of the first kind is considered. The corresponding result of Varma and Prasad [1] is generalized and perfected.

KEY WORDS AND PHRASES. Approximation, Hermite Interpolation, L_p -Norm.

1991 AMS SUBJECT CLASSIFICATION CODE. 41A05, 41A10, 41A35.

1. INTRODUCTION.

Let $-1 < x_n < x_{n-1} < \dots < 1$ be the zeros of $T_n(x) = \cos n\theta$, ($\cos \theta = x$), the n th degree Tchebycheff polynomial of the first kind.

If $f \in C^1[-1, 1]$, then it is known that a Hermite interpolation $H_n^*(f, x)$ of degree $\leq 2n-1$ which satisfies the conditions

$$H_n^*(f, x_k) = f(x_k) \text{ and } H_n^{*'}(f, x_k) = f'(x_k) \quad k = 1, \dots, n$$

is given by

$$H_n^*(f, x) = \sum_{k=1}^n f(x_k) h_k(x) + \sum_{k=1}^n f'(x_k) \sigma_k(x) \quad (1.1)$$

where

$$h_k(x) = (1 - xx_k) \left(\frac{T_n(x)}{n(x - x_k)} \right)^2 \geq 0, \quad \sum_{k=1}^n h_k(x) \equiv 1 \quad (1.2)$$

$$\sigma_k(x) = (x - x_k) l_k^2(x), \quad l_k(x) = \frac{T_n(x)}{T_n'(x_k)(x - x_k)} \quad (1.3)$$

Concerning the polynomial $H_n^*(f, x)$, Varma and Prasad [1] proved the following:

THEOREM A. Let $f \in C^1[-1, 1]$, then we have

$$\left(\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |H_n^*(f, x) - f(x)|^2 dx \right)^{1/2} \leq c n^{-1} E_{2n-2}(f'), \quad (1.4)$$

where $E_{2n-2}(f')$ is the best approximation to $f'(x)$ by polynomials of degree at most $2n-2$ and c is a positive absolute constant.

Naturally, one raises the problem that if there is similar result of (1.4) in L_p ($p > 0$) norm. Here we give an affirmative answer for the above problem, we shall prove the following:

THEOREM 1. Let $f \in C^1[-1, 1]$, then we have

$$\left(\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |H_n^*(f, x) - f(x)|^p dx \right)^{1/p} \leq c n^{-1} E_{2n-2}(f') \quad (1.5)$$

Therefore the corresponding result of [1] is generalized and perfected.

2. LEMMAS AND THE PROOF OF THEOREM 1.

At first, we state and prove several lemmas.

LEMMA 1 (Féjer [2]). If

$$\sum_{k=1}^n l_k^2(x) \leq 2, \quad (2.1)$$

therefore it follows that

$$\sum_{k=1}^n |l_k^2(x)|^r \leq c \quad r = 3, 4, \dots \quad (2.2)$$

LEMMA 2. Let k be even and $y_1, y_2, \dots, y_k$ be distinct integers between 1 and n , then we have ($k = 2m$)

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \sigma_{\gamma_1}(x) \sigma_{\gamma_2}(x) \cdots \sigma_{\gamma_k}(x) dx = 0 \quad (2.3)$$

PROOF. Since

$$\begin{aligned} \cos^{4m-1} n\theta &= \frac{1}{2^{2(2m-1)}} \sum_{j=0}^{2m-1} \binom{4m}{j} \cos(4m-2j-1)n\theta \\ &= \sum_{i \geq n}^{(4m-1)n} \mu_i \cos i\theta = \sum_{i \geq n}^{(4m-1)n} \mu_i T_i(x) \end{aligned} \quad (2.4)$$

and

$$\frac{T_n(x)}{(x-x_{\gamma_1}) \cdots (x-x_{\gamma_k})} = q_{n-2m}(x) \quad (2.5)$$

where $q_{n-2m}(x)$ is a polynomial of degree $\leq n-2m$.

On using these ideas together with orthogonality of Tchebycheff polynomials, we obtain

$$\begin{aligned} &\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \sigma_{\gamma_1}(x) \sigma_{\gamma_2}(x) \cdots \sigma_{\gamma_k}(x) dx \\ &= \frac{1}{[T'_n(x_{\gamma_1}) \cdots T'_n(x_{\gamma_k})]^2} \sum_{i \geq n}^{(4m-1)n} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} T_i(x) q_{n-2m}(x) dx = 0 \end{aligned}$$

This proves Lemma 2.

To prove Theorem 1 in the general case, we again follow the method of Erdős and Feldheim [3], it is enough to prove for even values of p only. To illustrate the method we limit for the case $p = 4$. For arbitrary fixed even p the proof is similar. Let $s_{2n-1}(x)$ be the polynomial of best approximation to $f(x)$ by the polynomials of degree $\leq 2n-1$. One can easily see that for $-1 \leq x \leq 1$:

$$H_n^*(f, x) - f(x) = H_n^*(f - s_{2n-1}, x) + s_{2n-1}(x) - f(x) \quad (2.6)$$

One notes that

$$|a + b|^p \leq c(p)(|a|^p + |b|^p) \quad (2.7)$$

where $c(p)$ is a constant of dependent of p only.

$$\begin{aligned} &\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |H_n^*(f, x) - f(x)|^4 dx \\ &\leq c \left[\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} H_n^{*4}(f - s_{2n-1}, x) dx + \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} (s_{2n-1}(x) - f(x))^4 dx \right] \\ &\leq c(I_1 + I_2) \end{aligned} \quad (2.8)$$

From the definition of $s_{2n-1}(x)$ we have

$$|s_{2n-1}(x) - f(x)| \leq E_{2n-1}(f) \quad (2.9)$$

where $E_{2n-1}(f)$ is the best approximation of $f(x)$.

From (2.9) we have

$$I_2 \leq \pi E_{2n-1}^4(f) \quad (2.10)$$

On using (1.1) and (2.7) we have

$$\begin{aligned} I_1 &\leq c \left\{ \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left[\sum_{k=1}^n (f(x_k) - s_{2n-1}(x_k)) h_k(x) \right]^4 dx \right. \\ &\quad \left. + \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left[\sum_{k=1}^n (f'(x_k) - s'_{2n-1}(x_k)) \sigma_k(x) \right]^4 dx \right\} \\ &= c(I_3 + I_4) \end{aligned} \quad (2.11)$$

Now from (1.2) and (2.9) it follows that

$$\begin{aligned} I_3 &\leq \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left[\sum_{k=1}^n (f(x_k) - s_{2n-1}(x_k)) h_k(x) \right]^4 dx \\ &\leq \pi E_{2n-1}^4(f) \end{aligned} \quad (2.12)$$

Let $\Delta_k = f'(x_k) - s'_{2n-1}(x_k) \quad k = 1, \dots, n$

One notes that

$$\begin{aligned} \left[\sum_{k=1}^n \Delta_k \sigma_k(x) \right]^4 &= \sum_{k \neq 1}^n \Delta_k^4 \sigma_k^4(x) + \sum_{k \neq j} \Delta_k^3 \Delta_j(x) \sigma_k^3(x) \sigma_j(x) \\ &\quad + \sum_{k \neq j \neq i} \Delta_k^2 \Delta_j \Delta_i \sigma_k^2(x) \sigma_j(x) \sigma_i(x) + \sum_{k \neq j \neq i \neq s} \Delta_k \Delta_j \Delta_i \Delta_s \sigma_k(x) \sigma_j(x) \sigma_i(x) \sigma_s(x) \\ &\quad + \sum_{k \neq j} \Delta_k^2 \Delta_j^2 \sigma_j^2(x) \sigma_j^2(x) = L_1(x) + L_2(x) + L_3(x) + L_4(x) + L_5(x) \end{aligned} \quad (2.13)$$

One notes also that

$$|T_n(x)| \leq 1 \quad (2.14)$$

$$1 - x_k^2 |\Delta_k| \leq 40 E_{2n-2}(f') \quad (\text{See [4]}) \quad (2.15)$$

and

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} l_k(x) l_j(x) dx = \begin{cases} 0 & k \neq j \\ \frac{\pi}{n} & k = j \end{cases} \quad (2.16)$$

Therefore from (1.3), (2.2) and (2.14)-(2.16) we have

$$\begin{aligned} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} L_1(x) dx &= \sum_{k=1}^n \frac{\Delta_k^4}{T_n^4(x_k)} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} T_n^4(x) l_k^4(x) dx \\ &\leq \frac{2}{n^4} \sum_{k=1}^n \left(\sqrt{1-x_k^2} |\Delta_k| \right)^4 \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} l_k^2(x) dx \leq c \frac{E_{2n-2}^4(f')}{n^4} \end{aligned} \quad (2.17)$$

One notes that

$$L_2(x) = \left(\sum_{k=1}^n \Delta_k^3 \sigma_k^3(x) \right) \left(\sum_{k=1}^n \Delta_k \sigma_k(x) \right) - L_1(x)$$

Using (1.3), (2.2), (2.14)-(2.17) and the Cauchy inequality we have

$$\begin{aligned}
& \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |L_2(x)| dx \\
& \leq \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^n \frac{(\sqrt{1-x_k^2} |\Delta_k|)^3}{n^3} |T_n^3(x)| |l_k^3(x)| \\
& \quad + \left| \sum_{k=1}^n \frac{(-1)^{k-1} \sqrt{1-x_k^2} \Delta_k}{n} T_n(x) l_k(x) \right| dx + c \frac{E_{2n-2}^4(f')}{n^4} \\
& \leq c \frac{E_{2n-2}^3(f')}{n^4} \\
& \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left| \sum_{k=1}^n (-1)^{k-1} \sqrt{1-x_k^2} \Delta_k l_k(x) \right| dx + c \frac{E_{2n-2}^4(f')}{n^4} \\
& \leq \frac{E_{2n-2}^3(f')}{n^4} \left(\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left[\sum_{k=1}^n (-1)^{k-1} \sqrt{1-x_k^2} \Delta_k l_k(x) \right]^2 dx \right)^{1/2} + c \frac{E_{2n-2}^4(f')}{n^4} \\
& \leq c \frac{E_{2n-2}^4(f')}{n^4}
\end{aligned} \tag{2.18}$$

From (1.3), (2.1), (2.15) and the estimation of $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |L_2(x)| dx$, we note that

$$L_3(x) = \left(\sum_{k=1}^n \Delta_k^2 \sigma_k^2(x) \right) \left[\left(\sum_{k=1}^n \Delta_k \sigma_k(x) \right)^2 - \sum_{j=1}^n \Delta_j^2 \sigma_j^2(x) \right] - L_2(x)$$

Thus we have also that

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |L_3(x)| dx \leq c \frac{E_{2n-2}^4(f')}{n^4} \tag{2.19}$$

Using Lemma 2 we have

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} L_4(x) dx = 0 \tag{2.20}$$

One notes that

$$L_5(x) = \left(\sum_{k=1}^n \Delta_k^2 \sigma_k^2(x) \right)^2 - L_1(x)$$

and similar to estimation of $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |L_3(x)| dx$ we have

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |L_5(x)| dx \leq c \frac{E_{2n-2}^4(f')}{n^4} \tag{2.21}$$

From (2.17) - (2.21) we have

$$I_4 \leq c \frac{E_{2n-2}^4(f')}{n^4} \tag{2.22}$$

Combining (2.11), (2.12) and (2.22) we obtain

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} |H_n^*(f, x) - f(x)|^4 dx \leq c \frac{E_{2n-2}^4(f')}{n^4} \tag{2.23}$$

This proves Theorem 1.

3. REMARKS.

1. Concerning quasi-Hermite interpolation [5] based on the zeros of Tchebycheff polynomial of the second kind, there is similar result in Theorem 1.

2. For almost-Hermite interpolation [6] based on the zeros of $(1-x)J_n^{(1/2, -1/2)}(x)$ (or $(1+x)J_n^{(-1/2, 1/2)}(x)$) (where $J_n^{(\alpha, \beta)}(x)$ be the Jacobi polynomial), there is similar result of Theorem 1 also.

Here we omit the details.

REFERENCES

1. VARMA, A.K. and PRASAD, J., An analogue of a problem of P. Erdos and E. Feldheim on L_p convergence of interpolatory processes, J. Approx. Th., **56** (1989), 225-240.
2. SZEGO, G., Orthogonal polynomials, A.M.S. Coll. Publ., New York, 1978.
3. ERDOS, P. and FELDHEIM, E., Sur le mode de convergence pour l'interpolation de Lagrange, C.R. Acad. Sci. Paris, **203**, (1937), 913-915.
4. RIVLIN, T.J., Introduction to the approximation of functions, Ginn (Blaisdell), Boston, 1969.
5. SZASZZ, P., On quasi-Hermite-Fejér interpolation, Acat Math. Acad. Sic. Hung., **10** (1959), 413-439.
6. GONSKA, H.H., On almost-Hermite-Fejér, interpolation: pointwise estimates, Bull. Austral. Math. Soc., **25** (1982), 405-423.