REMARKS ON DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT. We prove that a semiprime ring R must be commutative if it admits a derivation d such that (i) $xy + d(xy) = yx + d(yx)$ for all x, y in R , or (ii) $xy - d(xy) = yx - d(yx)$ for all x, y in R . In the event that R is prime, (i) or (ii) need only be assumed for all x, y in some nonzero ideal of R .

KEY WORDS AND PHRASES. Derivation, semiprime ring, prime ring, commutative, central ideal, integral domain, direct sum.

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1. INTRODUCTION.

In the past fifteen years, there has been an ongoing interest in derivations on prime or semiprime rings; and many of the results have involved commutativity. (See [1] for a partial bibliography.) In this brief note, we explore the commutativity implications of the existence on R of a derivation d satisfying the following:

(*) there exists a nonzero ideal K of R such that either $xy + d(xy) = yx + d(yx)$ for all x, y in K , or $xy - d(xy) = yx - d(yx)$ for all x, y in K .

2. THE PRINCIPAL RESULTS.

Our principal results in this note are

THEOREM 1. If R is any prime ring admitting a derivation d satisfying (*), then R is commutative.

THEOREM 2. Let R be a semiprime ring admitting a derivation d for which either $xy + d(xy) = yx + d(yx)$ for all x, y in R or $xy - d(xy) = yx - d(yx)$ for all x, y in R . Then R is commutative.

In fact, both of these theorems are consequences of a third theorem, which is reminiscent of the results in [1].

THEOREM 3. If R is a semiprime ring admitting a derivation d satisfying (*), the K is a central ideal.

3. PROOFS.

The proof of Theorem 3 hinges on the following lemma.

LEMMA 1. Let R be a semiprime ring and I a nonzero ideal of R . If z in R centralizes the set $[I, I]$, then z centralizes I .

PROOF. Let z centralizes $[I, I]$. Then for all x, y in I , we have $z[x, y] = [x, y]z$, which can be rewritten as $zx[x, y] = x[x, y]z$; hence $[z, x][x, y] = 0$ for all x, y in I . Replacing y by yz , we get $[z, x]I[z, x] = \{0\}$. Since I is an ideal, it follows that, $[z, x]IR[z, x]I = \{0\} = I[z, x]RI[z, x]$, so that $[z, x]I = I[z, x] = \{0\}$. Thus, $[[z, x], x] = 0$ for all x in I ; and by Theorem 3 of [2], z centralizes I .

For ease of reference, we include a second lemma, which is well-known.

LEMMA 2. (a) If R is a prime ring with a nonzero central ideal, then R is commutative.

(b) If R is a semiprime ring, the center of a nonzero ideal is contained in the center of R .

PROOF OF THEOREM 3. We suppose first that

$$xy + d(xy) = yx + d(yx) \text{ for all } x, y \text{ in } K, \quad (1)$$

which can be rewritten as

$$[x, y] = -d([x, y]) \text{ for all } x, y \text{ in } K. \quad (2)$$

Now for all x, y, z in K , we have $[x, y]z + d([x, y]z) = z[x, y] + d(z[x, y])$, which yields

$$[x, y]z + d([x, y])z + [x, y]d(z) = z[x, y] + d(z)[x, y] + zd([x, y]);$$

and applying (2) we conclude that

$$[x, y]d(z) = d(z)[x, y] \text{ for all } x, y, z \text{ in } K. \quad (3)$$

By Lemma 1, we see that $d(K)$ centralizes K ; and it follows from (1) that $[x, y]$ is in the center of K for all x, y in K . Another application of Lemma 1 shows that the ideal K is commutative; hence by Lemma 2(b), K is in the center of R . In the event that $xy - d(xy) = yx - d(yx)$ for all x, y in K , it is equally easy to establish (3), therefore our proof is complete.

Theorem 2 is immediate from Theorem 3, and Theorem 1 follows from Theorem 3 and Lemma 2(a).

We remark, in conclusion, that under the hypotheses of Theorem 3 we cannot hope to prove commutativity of R . Consider $R = R_1 \oplus R_2$, where R_1 is an integral domain, R_2 is a prime ring which is not commutative, and d is the "direct sum" of derivations on the summands R_1 and R_2 .

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