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INTERPOLATION BY NATURAL CUBIC SPLINE

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ABSTRACT. From the result in [1] it follows that there is a unique quadratic spline which bounds the same area as that of the function. The matching of the area for the cubic spline does not follow from the corresponding result proved in [2]. We obtain cubic splines which preserve the area of the function.

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1. INTRODUCTION AND NOTATION.

Let $\Delta = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ be a partition of [0,1] such that $x_i - x_{i-1} = h$, $i = 1,2,\ldots,n$. We write π_m for the class of all algebraic real polynomials of degree m or less. Let

$$S(m,\Delta) = \{s(x):s(x) \in \pi_m,\ s(x) \in C^{m-1}[0,1],\ x \in [x_{i-1}, x_i], i = 1,2,\ldots n\}$$

denote the class of all polynomial splines, where m is a positive interger.

Approximation of function by cubic spline interpolation has been studied by [3],[4]. Further studies in this direction are due to Sharma and Tzimbalario [1], deBoor [5], Schoenberg [6] based on finding the splines which have the same integral mean as the function has with respect to certain measure. The following theorem is due to Dikshit [2].

THEOREM 1. Let $f \in C^2[0,1]$ be a 1-periodic locally integrable function with respect to a non-negative measure $d\mu$ satisfying $\mu(x+h)-\mu(x)=$ constant. Suppose further that either

$$\int_0^h \alpha(x)\, d\mu > 0 \quad \text{or} \quad \int_0^h \alpha(h-x)\, d\mu > 0, \tag{1.1}$$

where $\alpha(x)= 3x^3-6hx^2+h^3$. Then there exists a unique $s(x)\in S(3,\Delta)$ satisfying the conditions:

$$\int_{x_{i-1}}^{x_i} \{f(x)-s(x)\}\, d\mu = 0, \quad i = 1,2,\ldots n; \tag{1.2}$$

and $$s^r(0) = s^r(1), \quad r = 0,1,2. \tag{1.3}$$

One of the important cases of the associated measure function $\mu(x)$ of Theorem 1 is

when it is linear, i.e. $\mu(x) = x$. This is not included, as in this case the value of the integrals in (1.1) is $-h^4/4$. This special case is interesting because the function and the spline bound the same area. We can see from Rolle's theorem that this condition forces the spline to match a continuous function f at least at one point of each subinterval.

2. We prove the following:

THEOREM 2. Let $f \in L[0,1]$. Then there exists a unique spline $s(x) \in S(3,\Delta)$ which bounds the same area as the function does, precisely,

$$\int_{x_{i-1}}^{x_i} f(x)dx = \int_{x_{i-1}}^{x_i} s(x)dx, \qquad i = 1,2, \ldots, n, \tag{2.1}$$

if $s''(0) = s''(1) = 0$ and $h(s''(1-h) - s''(h)) = 24\, S'(0)$.

We need the following lemma for the proof of Theorem 2.

LEMMA 1. Let $C_n(s,0)$ and $C_n'(0,0)$, $n \geq 4$, be the following n-square quadruple diagonal matrices of nonnegative real numbers s,r,p,q:

$$C_n(s,0) = \begin{bmatrix} q & p & 0 & 0.. & 0 & 0 & 0 & s \\ r & q & p & 0.. & 0 & 0 & 0 & 0 \\ s & r & q & p.. & 0 & 0 & 0 & 0 \\ . & . & . & ... & . & . & . & . \\ . & . & .. & . & . & . & . & . \\ 0 & 0 & 0 & 0.. & s & r & q & p \\ 0 & 0 & 0 & 0.. & 0 & s & r & q \end{bmatrix}; \quad C_n'(0,0) = \begin{bmatrix} r & q & p & 0 & 0.. & 0 & 0 & 0 \\ s & r & q & p & o.. & 0 & 0 & 0 \\ 0 & s & r & q & p.. & 0 & 0 & 0 \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ 0 & 0 & 0 & 0 & 0.. & s & r & q \\ 0 & 0 & 0 & 0 & 0.. & 0 & s & r \end{bmatrix}.$$

Then $C_n(s,0)$ is non-singular for odd n, if $q^2 - 4pr > 0$ and $r^2 - 4sq > 0$. The result also holds for even n if, in addition, $D_n(0,0) - sD_{n-1}'(0,0) > 0$, where $D_n(0,0)$ and $D_n'(0,0)$ are the determinants of $C_n(0,0)$ and $C_n'(0,0)$ respectively.

PROOF. For the sake of convenience, we write

$Q = q^2 - 4pr$, $R = r^2 - 4sq$, $u = 2[(n-j)/2]+1$, $v = n-2[(3j+1)/2]$, $w=(n-3j)-(1+(-1)^j)/2$

and set, for $k > 0$,

$$f(n,j,k) = f_1(n,j,k) = \frac{u(u-2)\ldots(u-2k+2)}{v(v-2)\ldots(v-2k+2)} \qquad \text{for odd } n;$$

$$f(n,j,k) = f_2(n,j.k) = \frac{u(u-2)\ldots(u-2k+2)}{w(w-2)\ldots(w-2k+2)} \qquad \text{for even } n;$$

and $f(n,j,0) = f_1(n,j,0) = f_2(n,j,0) = 1$.

$[y]$ denotes greatest integer less than or equal to y.

We first prove that

$$D_n(0,0) = \sum_{t=0}^{[n/3]} \alpha(n,t), \tag{2.2}$$

where

$$\alpha(n,0)=\begin{cases} q\sum_{j=0}^{(n-1)/2}\binom{n-j}{j} Q^{((n-1)/2)-j}(pr)^j, & \text{for odd } n;\\ Q^{n/2}+\sum_{j=1}^{n/2}\frac{n+1}{j}\binom{n-j}{j-1} Q^{(n/2)-j}(pr)^j, & \text{for even } n;\end{cases}$$

and for $t > 0$,

$$\alpha(n,t)=(p^2s)^t\, q^{(1-(-1)^{n+t})/2}\sum_{k=0}^{[(n-3t)/2]} f(n,t,k)\binom{k+t}{t}\binom{n-2t-k}{k+t}Q^{[(n-3t)/2]-k}(pr)^k .$$

We can see directly that (2.2) holds for $n = 4,5,6$. Now we assume that (2.2) is true upto n-th place and n is even. From the matrix $C_n(0,0)$ it can be seen that

$$D_{n+1}(0,0) = q\, D_n(0,0) - pr\, D_{n-1}(0,0) + p^2s\, D_{n-2}(0,0). \tag{2.3}$$

Since (2.2) is true for n, n-1, n-2, we have

$$q\, D_n(0,0) - pr\, D_{n-1}(0,0)+ p^2s\, D_{n-2}(0,0)$$

$$= q\,\alpha(n,0)-pr\,\alpha(n-1,0)+\sum_{t=1}^{[(n-2)/3]}\{q\,\alpha(n,t)-pr\,\alpha(n-1,t)+(p^2s)\,\alpha(n-2,t-1)\} + X(n), \tag{2.4}$$

where the last term $X(n)$ is

$$X(n) = q\,\alpha(n,[n/3]) - pr\,\alpha(n-1,[(n-1)/3]) + p^2s\,\alpha(n-2,[(n-2)/3]) \quad \text{for } n= 6m+4,$$

$$X(n) = q\,\alpha(n,[n/3]) + p^2s\,\alpha(n-2,[(n-2)/3]) \quad \text{for } n=6m,$$

$$X(n) = p^2s\,\alpha(n-2,[(n-2)/3]) \quad \text{for } n = 6m+2,$$

with some positive integral values of m.

On simplification, we obtain

$$q\,\alpha(n,0)-pr\,\alpha(n-1,0) = \alpha(n+1,0). \tag{2.5}$$

Now, we consider the following sum for relevant odd values of t in the summation of (2.4). Since by the assumption n+t is odd, the exponent of q in $\alpha(n,t)$ is 1. By writing $q^2 = Q+4pr$, we have

$$q\,\alpha(n,t)-pr\,\alpha(n-1,t)+(p^2s)\,\alpha(n-2,t-1)$$

$$= Q\,\alpha(n,t)/q + 4pr\,\alpha(n,t)/q - pr\,\alpha(n-1,t) + p^2s\,\alpha(n-2,t-1)$$

$$= (p^2s)^t\,\{f_2(n,t,0)\binom{n-2t}{t}Q^{[(n-3t)/2]+1} + f_2(n-2,t-1,0)\binom{n-2t}{t-1}Q^{[(n-3t)/2)]+1}\}$$

$$+ (p^2s)^t\Big[\sum_{k=1}^{[(n-3t)/2]} f_2(n,t,k)\binom{k+t}{t}\binom{n-2t-k}{k+t}Q^{[(n-3t)/2]-k+1}(pr)^k$$

$$+ 4\sum_{k=1}^{[(n-3t)/2]+1} f_2(n,t,k-1)\binom{k+t-1}{t}\binom{n-2t-k+1}{k+t-1}Q^{[(n-3t)/2]-k+1}(pr)^k$$

$$-\sum_{k=1}^{[(n-3t)/2]+1} f_1(n-1,t,k-1)\binom{k+t-1}{t}\binom{n-2t-k}{k+t-1}Q^{[(n-3t)/2]-k+1}(pr)^k$$

$$+\sum_{k=1}^{[(n-3t)/2]+1} f_2(n-2,t-1,k)\binom{k+t-1}{t-1}\binom{n-2t-k}{k+t-1}Q^{[(n-3t)/2]-k+1}(pr)^k\Big], \tag{2.6}$$

by taking the first terms from the first and last summations and changing the summing index in second and third summations in view of the fact that $[(n-3t)/2] = [(n-3t-1)/2]$. We first observe that

$$\begin{aligned} f_2(n,t,k) &= \{(n-t-2k+2)/(n-t+2)\}\, f_1(n+1,t,k) \\ f_2(n,t,k-1) &= \{(n-3t-2k+2)/(n-t+2)\}\, f_1(n+1,t,k) \\ f_1(n-1,t,k-1) &= \{(n-3t)/(n-t+2)\}\, f_1(n+1,t,k) \end{aligned} \tag{2.7}$$

and $\quad f_2(n-2,t-1,k) = \{(n-t-2k+2)/(n-t+2)\}\, f_1(n+1,t,k)$.

In view of these relations, we first combine $((n-3t-1)/2)$ terms of the four summations and then combine the last terms of the second, third and fourth summations. Thus, the expression in (2.6) can be written as

$$\begin{aligned} &(p^2s)^t \binom{n+1-2t}{t} Q^{[(n-3t)/2]+1} \\ &+ (p^2s)^t \sum_{k=1}^{[(n-3t)/2]} f_1(n+1,t,k) \binom{k+t}{t}\binom{n+1-2t-k}{k+t} Q^{[(n-3t/2]-k+1} (pr)^k \\ &+ (p^2s)^t \left[f_1(n+1,t,k)\binom{k+t}{t}\binom{n+1-2t-k}{k+t} Q^{[(n-3t)/2]-k+1} (pr)^k \right]_{k=[(n-3t)/2]+1} \end{aligned} \tag{2.8}$$

This proves that, for odd t,

$$q\,\alpha(n,t)-pr\,\alpha(n-1,t) + p^2s\,\alpha(n-2,t-1) =$$

$$(p^2s)^t\, q^{(1-(-1)^{n+1+t})/2} \sum_{k=0}^{[(n+1-3t)]/2} f_1(n+1,t,k) \binom{k+t}{t}\binom{n+1-2t-k}{k+t} Q^{[(n+1-3t)/2]-k} (pr)^k. \tag{2.9}$$

From the proof for (2.9) we can see that it continues to hold for even values of t also.

Lastly, we consider the term $X(n)$ for $n = 6m+4$. We have

$$\begin{aligned} X(n) &= q\,\alpha(n,[n/3]) - pr\,\alpha(n-1,[(n-1)/3]) + p^2s\,\alpha(n-2,[(n-2)/3]) \\ &= \left[(p^2s)^t(Q+4pr)\binom{n-2t}{t}\right]_{t=[n/3]} - pr\left[(p^2s)^t\binom{n-2t-1}{t}\right]_{t=[(n-1)/3]} \\ &+ (p^2s)\left[(p^2s)^t\binom{n-2t-2}{t} Q^{[(n-3t-2)/2]} + f_2(n-2,t,1)\binom{t+1}{t}\binom{n-2t-3}{t+1} pr \right]_{t=[(n-2)/3]} \\ &= (p^2s)^t q^{(1-(-1)^{n+1+t})/2} \sum_{k=0}^{[(n+1-3t)/2]} f_1(n+1,t,k)\binom{k+t}{t}\binom{n+1-2t-k}{t+k} Q^{[(n+1-3t)/2]-k}(pr)^k, \end{aligned} \tag{2.10}$$

where $t=[(n+1)/3]$. Similarly, it can be seen that the relation (2.10) holds for $n = 6m$ and $n=6m+2$. Now on combining (2.5),(2.9) and (2.10) we get

$$qD_n(0,0) - prD_{n-1}(0,0) + p^2sD_{n-2}(0,0) = \sum_{t=0}^{[(n+1)/3]} \alpha(n+1,t). \tag{2.11}$$

If we start with odd vaues of n instead of even values we get the relation (2,11).We can obtain in a similar way that

$$D_n'(0,0) = \sum_{t=0}^{[n/3]} \beta(n,t), \tag{2.12}$$

where

$$\beta(n,0)= \begin{cases} r \sum_{j=0}^{(n-1)/2} \binom{n-j}{j} R^{((n-1)/2)-j} (sq)^j, & \text{for odd } n; \\ R^{n/2} + \sum_{j=1}^{n/2} \frac{n+1}{j}\binom{n-j}{j-1} R^{(n/2)-j} (sq)^j, & \text{for even } n; \end{cases}$$

and for $t > 0$,

$$\beta(n,t) = (s^2p)^t \; r^{(1-(-1)^{n+t})/2} \sum_{k=0}^{[(n-3t)/2]} f(n,t,k) \binom{k+t}{t}\binom{n-2t-k}{k+t} R^{[(n-3t)/2]-k} (sq)^k .$$

Now Lemma 1 follows from the following relation:

$$D_n(s,0) = D_n(0,0) + (-1)^{n-1} s\, D_{n-1}'(0,0). \tag{2.13}$$

3. PROOF OF THE THEOREM 2. For $i = 1,2,\ldots,n$, we set

$$F_i = \int_{x_{i-1}}^{x_i} f(x)\,dx, \qquad s''(x_i) = M_i \quad \text{and} \quad s''(x_0) = M_0. \tag{3.1}$$

We see that for $x \in [x_{i-1}, x_i]$

$$s'(x) = -(1/2h)M_{i-1}(x_i-x)^2 + (1/2h)M_i(x-x_{i-1})^2 + c_i , \tag{3.2}$$

where $i = 1,2,\ldots,n$ and c_i's are the constants. As $s'(x) \in C[0,1]$, we have

$$M_i h = c_{i+1} - c_i, \quad i = 1,2,\ldots,n-1. \tag{3.3}$$

From (3.2) we get

$$s(x) = (1/6h)[M_{i-1}(x_i-x)^3 + M_i(x-x_{i-1})^3] -(1/2)c_i[(x_i-x)-(x-x_{i-1})]+d_i. \tag{3.4}$$

Continuity of $s(x)$ yields

$$h(c_i + c_{i+1}) = 2(d_{i+1}-d_i), \quad i = 1,2,\ldots,n-1. \tag{3.5}$$

Applying the interpolatory condition (2.1), we obtain

$$6F_i = (h^3/4)\,[M_{i-1} + M_i] + 6d_i h, \quad i = 1,2,\ldots,n. \tag{3.6}$$

Equations (3.5) and (3.6) give

$$(h^3/4)[M_{i+1} - M_{i-1}]+3h^2(c_i+c_{i+1}) = 6(F_{i+1}-F_i), \quad i=1,2,\ldots,n-1 . \tag{3.7}$$

Now using (3.3), we get

$$M_{i+1}h^3/4 - 3M_i h^3 - M_{i-1}h^3/4 + 6h^2c_{i+1} = 6(F_{i+1}-F_i).$$

Finally we get the system of linear equations,for $i = 2,\ldots,n-1$,as

$$M_{i+1}h/4 + 11M_i h/4 + 11M_{i-1}h/4 + M_{i-2}h/4 = 6(F_{i+1}-2F_i+F_{i-1})h^{-2}. \tag{3.8}$$

From (3.3) and (3.7)and that $M_0 = 0$, we obtain

$$24c_1 = 24h^{-2}\,(F_2 - F_1) - 12M_1 h - M_2 h.$$

Hence the boundary condition gives

$$M_2h/4 + 11M_1h/4 + M_{n-1}h/4 = 6(F_2 - F_1)h^{-2}. \tag{3.9}$$

Thus (3.8) and (3.9) give n-1 equations. Since by the boundary conditions $M_n = M_0 = 0$, the spline exists if the system of linear equations in M_i's gives unique solutions. We take $q = 11h/4$, $r = 11h/4$, $s = h/4$ and $p = h/4$. We have $Q > 0$ and $R > 0$. Consequently by Lemma 1, the solutions exist for odd n. In order to prove that it is also true for even n, it is sufficient to show that

$$D_n(s,0) = D_n(0,0) - s\, D'_{n-1}(0,0) > 0.$$

From (2.2) and (2.12) we get

$$D_n(s,0) = \alpha(n,0) - s\,\beta(n-1,0) + \sum_{j=1}^{[(n-1)/3]} \{\alpha(n,j) - s\,\beta(n-1,j)\} + \alpha(n,[n/3]). \tag{3.10}$$

We have

$$\alpha(n,0) - s\,\beta(n-1,0) = Q^{n/2} + \sum_{k=1}^{n/2} \left\{\frac{(n+1)}{k} - 1\right\}\binom{n-k}{k-1} Q^{(n/2)-k}\,(pr)^k > 0. \tag{3.11}$$

In order to show that the remaining part is also positive, we first observe from (2.7) that

$$f_2(n,t,k-1) = \frac{(n-3t-2k+2)}{n-3t}\, f_1(n-1,t,k-1)\,.$$

Consequently,

$$\alpha(n,j) - s\,\beta(n-1,j)$$

$$= (p^2 s)^j \sum_{k=0}^{[(n-3j)/2]} f_1(n-1,j,k)\binom{k+j}{j}\binom{n-2j-k-1}{k+j}\left\{\frac{q(n-2j-k)}{n-3j} - s\right\} Q^{(n-3j-1)/2-k}\,(pr)^k.$$

In a similar way we can show that the inner sum is positive for even values of t. This completes proof of the theorem.

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