ON WEIGHTS WHICH ADMIT THE REPRODUCING KERNEL OF BERGMAN TYPE

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ABSTRACT. In this paper we consider (1) the weights of integration for which the reproducing kernel of the Bergman type can be defined, i.e., the admissible weights, and (2) the kernels defined by such weights. It is verified that the weighted Bergman kernel has the analogous properties as the classical one. We prove several sufficient conditions and necessary and sufficient conditions for a weight to be an admissible weight. We give also an example of a weight which is not of this class. As a positive example we consider the weight $\mu(z) = (\operatorname{Im} z)^2$ defined on the unit disk in $\mathbb{C}$.

1. INTRODUCTION.

In this paper we are concerned with the weights of integration, the so called admissible weights or a-weights for short for which the Bergman reproducing kernel can be defined. Recently reproducing kernels of Bergman type depending on weights of integration have been used to define the quantization of classical states in holomorphic models of quantum field theory (Odzijewicz [1]). Earlier they have appeared in studies of wave and Dirac equations (Jakobsen and Vergne [2]). Functions of this type have been also considered in consequence of many mathematical problems (Burbea and Masani [3] and Mazur [4]). In (Winiarski [5]) the concept of the a-weight was introduced. The main purpose of the presented study is to give a more detailed characterization of a-weights.

In Section 2 we introduce the new definition of the a-weight. We introduce also the Bergman function defined by such a weight, Definition 2.1 and 2.2. Next we show several basic properties of the Bergman function, Theorem 2.1 and Proposition 2.1. Theorem 2.2 contains some necessary and sufficient conditions for a weight to be an a-weight. Point (iv) of this theorem shows that Definition 2.1 is equivalent to the definition of the a-weight given in [5]. Other more effective sufficient conditions are given in Section 3, Theorem 3.1, and Corollary 3.1 and 3.2. In Section 4 we present the example of a weight which is not an a-weight.

If μ is a weight, i.e., a real-valued, almost everywhere positive Lebesque measurable function on an open set $D \subset \mathbb{C}^n$, then define a set Z_μ as follows: $z \in Z_\mu$ iff for any neighborhood U of z

$$\operatorname*{ess\,inf}_{z \in U} \mu(z) = 0.$$

All Bergman functions considered to this time have been defined by weights for which $Z_\mu = \emptyset$ or the closure of Z_μ is a compact subset of D. In Section 5 we deal with the weight $\mu(z) = (\operatorname{Im} z)^2$ on the unit disc D in $\mathbb{C}$. In this case $Z_\mu = \{z \in D : \operatorname{Im} z = 0\}$ is a continuum which "comes to the

boundary" of D. By Corollary 3.2, μ is an a-weight. We find a complete orthonormal system in the space $L^2H(D,\mu)$ of all holomorphic μ-square integrable functions on D. It makes possible the uniform approximation of the Bergman kernel K_μ by polynomials on any compact subset of $D \times D$, see Theorem 2.1, (i).

Without any other explanations we use the following symbols: $\mathbf{N}$ - the set of natural numbers; $\mathbf{Z}$ - the set of integers; $\mathbf{Z}^+ := \mathbf{N} \cup \{0\}$; $\mathbf{R}$ - the set of reals, $\mathbf{C}$ the complex plane.

2.   ADMISSIBLE WEIGHTS AND REPRODUCING KERNELS.

Let D be an open nonempty set in $\mathbf{C}^n$ and let $W(D)$ be the set of weights on D, i.e., $W(D)$ is the set of all Lebesque measurable real-valued positive functions on D under the convention that we consider two weights as equivalent if they are equal almost everywhere with respect to the Lebesque measure on D. If $\mu \in W(D)$, we denote by $L^2(D,\mu)$ the space of all Lebesque measurable complex-valued μ-square integrable functions on D.

The space $L^2(D,\mu)$ is a separable complex Hilbert space with respect to the norm $\| \cdot \|_\mu$ given by the scalar product

$$< f \mid g >_\mu := \int_D \overline{f(z)}\, g(z)\, \mu(z)\, (dz)^{2n}, \quad f, g \in L^2(D,\mu).$$

The map $L^2(D,\mu) \ni f \to \sqrt{\mu} \cdot f \in L^2(D)$ is an isometry on $L^2(D,\mu)$ onto the space $L^2(D)$ of all complex-valued functions on D which are square integrable with respect to the Lebesque measure. Let $L^2H(D,\mu)$ be the set of all holomorphic functions from $L^2(D,\mu)$. Then $L^2H(D,\mu)$ is a linear subspace of $L^2(D,\mu)$ perhaps equal to $\{0\}$ called the μ-Bergman space over D (see [1]). For any $z \in D$ we define the evaluation functional E_z on $L^2H(D,\mu)$ by the formula

$$E_z f := f(z), \quad f \in L^2H(D,\mu). \tag{2.1}$$

DEFINITION 2.1.   A weight $\mu \in W(D)$ is called an admissible weight, an a-weight for short, if $L^2H(D,\mu)$ is a closed subspace of $L^2(D,\mu)$ and for any $z \in D$ the evaluation functional E_z is continuous on $L^2H(D,\mu)$. The set of all a-weights on D will be denoted by $AW(D)$.

It is well known that the characteristic function of the set D satisfies conditions of the above definition (Krantz [6] and Maurin [7]).

If $\mu \in AW(D)$ then by the Riesz representation theorem, for any $z \in D$ there exists a unique function $e_{z,\mu} \in L^2H(D,\mu)$ such that for each $f \in L^2H(D,\mu)$

$$E_z f = < e_{z,\mu} \mid f >_\mu = \int_D \overline{e_{z,\mu}(w)}\, f(w)\, \mu(w)\, (dw)^{2n} \tag{2.2}$$

DEFINITION 2.2.   The function $K_\mu: D \times D \to \mathbf{C}$ given by the formula

$$K_\mu(z,w) := \overline{e_{z,\mu}(w)}, \quad z, w \in D,$$

is called the μ-Bergman function of the set D (see [3]). Since the formula (2.2) can be written in the form

$$f(z) = \int_D z_\mu(z,w)\, f(w)\, \mu(w)\, (dw)^{2n} \tag{2.3}$$

the function K_μ is also called the μ-Bergman reproducing kernel of D.

Similarly as in the classical case we can show the following properties of K_μ (see [3] and [6]).

THEOREM 2.1. Let $\mu \in AW(D)$ and let K_μ be the μ-Bergman function. Then

(i) for any complete orthonormal system $\{\phi_k\}$ in $L^2H(D, \mu)$ and any $(z, w) \in D \times D$

$$K_\mu(z, w) = \sum_k \phi_k(z)\, \overline{\phi_k(w)}\, ; \tag{2.4}$$

(ii) for any $(z, w) \in D \times D$

$$K_\mu(w, z) = \overline{K_\mu(z, w)}\, ; \tag{2.5}$$

(iii) the function $K_\mu(z, w)$ is antiholomorphic in w and holomorphic in z;

(iv) K_μ is analytic in the real sense;

(v) if P_μ is the orthogonal projector on $L^2(D, \mu)$ upon $L^2H(D, \mu)$ then for every $f \in L^2(D, \mu)$ and each $z \in D$

$$[P_\mu f](z) = \int_D K_\mu(z, w)\, f(w)\, \mu(w)(dw)^{2n}$$

i.e., K_μ is the integral kernel of the operator P_μ.

PROOF. Let $\{\phi_k\}$ be a complete orthonormal system in $L^2H(D, \mu)$ and let for each $z \in D$

$$\overline{K_\mu(z, \cdot)} = e_{z, \mu} = \sum_k a_k(z)\phi_k \tag{2.7}$$

where $\{a_k(z)\}$ is the sequence of complex numbers. If $\{\phi_k\}$ is an infinite sequence, the series on the right hand side of (2.7) converges in $L^2H(D, \mu)$. Since the evaluation functionals are all continuous we obtain that for every $w \in D$

$$K_\mu(z, w) = \sum_k \overline{a_k(z)}\ \overline{\phi_k(w)}. \tag{2.8}$$

On the other hand for any k we have

$$\phi_k(z) = <e_{z, \mu} \,|\, \phi_k>_\mu = \sum_j \overline{a_j(z)} <\phi_j \,|\, \phi_k>_\mu = \overline{a_k(z)}, \tag{2.9}$$

see (2.2). Applying it to (2.8) we get (2.4). The point (ii) is an immediate consequence of (i).

By its very definition $\overline{K_\mu(z, w)}$ is holomorphic in w. Hence $K_\mu(z, w)$ is antiholomorphic in w and, by (ii), it is holomorphic in z.

In order to prove (iv) let us consider the function

$$D \times D^0 \ni (z, w) \to K_\mu^0(z, w) := K_\mu(z, \overline{w}) \in \mathbb{C},$$

where $D^0 = \{w \in \mathbb{C}^n \colon \overline{w} \in D\}$. From (iii) it follows that K_μ^0 is holomorphic in each variable separately. Then by the Hartogs theorem on separate analyticity, it is holomorphic on $D \times D^0$ (see [6]).

For the proof of (v) let $f \in L^2(D, \mu)$ and let $z \in D$. We have by (i)

$$\int_D K_\mu(z, w)\, f(w)\, \mu(w)(dw)^{2n} = <e_{z, \mu} \,|\, f>_\mu$$

$$= <\sum_k \overline{\phi_k(z)}\, \phi_k \,|\, f>_\mu = \sum_k \phi_k(z) <\phi_k \,|\, f>_\mu$$

$$= [\sum_k <\phi_k \,|\, f>_\mu \phi_k](z) = [P_\mu f](z),$$

where $\{\phi_k\}$ is an arbitrary complete orthonormal system in $L^2H(D, \mu)$. This completes the proof of our theorem.

PROPOSITION 2.1. Under the assumptions of Theorem 2.1. let $\{\phi_m\}$ be an orthonormal complete system in $L^2H(D, \mu)$. If $\{\phi_m\}$ is an infinite sequence then the series $\sum_{m=1}^{\infty} \phi_m(z)\overline{\phi_m(w)}$ converges to $K_\mu(z, w)$ uniformly on any compact set $M \subset D \times D$.

PROOF. We can assume that $M = X \times X$, where X is an arbitrary compact subset of D, i.e., for any compact set $M \subset D \times D$ there exists a compact set $X \subset D$ such that $M \subset X \times X$. From Theorem 2.1, (i) it follows that for any $z \in D$

$$\sum_{m=1}^{\infty} |\phi_m(z)|^2 = K_\mu(z, z) < \infty$$

i.e., $(\phi_m(z)) \in l^2$. Let $T_n(z) := \sum_{m=n}^{\infty} |\phi_m(z)|^2, z \in D$.
We have that (T_n) is a decreasing sequence of continuous functions on D, which converges to $f = 0$. By the Dini theorem (Maurin [8]), Chapter V, Section 4) we obtain that (T_n) converges uniformly on any compact set $X \subset D$. The uniform convergence of $\sum_{m=1}^{\infty} \phi_m(z)\overline{\phi_m(w)}$ on $X \times X$ follows now from the Schwarz inequality

$$\left| \sum_{m=1}^{\infty} \phi_m(z)\overline{\phi_m(w)} \right|^2 \leq T_n(z) T_n(w), \quad (z, w) \in X \times X.$$

Now we shall give the necessary and sufficient conditions for a weight $\mu \in W(D)$ to be an a-weight.

THEOREM 2.2. Let $\mu \in W(D)$. The following are equivalent:

(i) μ is an admissible weight;

(ii) for any compact set $X \subset D$ there exists a constant $C_X > 0$ such that for any $z \in X$ and each $f \in L^2H(D, \mu)$

$$|E_z f| \leq C_X \|f\|_\mu; \tag{2.10}$$

(iii) for any $z \in D$ there exists a compact set $Y \subset D$ which contains z and has the following property: for each $z' \in \partial Y$ there exists a neighbourhood $V_{z'}$ of z' in D and a constant $C_{z'} > 0$ such that for any $w \in V_{z'}$ and any $f \in L^2H(D, \mu)$

$$|E_w f| \leq C_{z'} \|f\|_\mu. \tag{2.11}$$

∂Y denotes the topological boundary of Y.

(iv) for any $z \in D$ there exists a neighborhood V_z of z in D and a constant $C_z > 0$ such that for any $w \in V_z$ and each $f \in L^2H(D, \mu)$

$$|E_w f| \leq C_z \|f\|_\mu. \tag{2.12}$$

PROOF. (i) $\Rightarrow$ (ii). Notice that, by Theorem 2.1, (i) and by the Riesz representation theorem, for any $z \in D$

$$\|E_z\|_\mu = \|K_\mu(z, \cdot)\|_\mu = \left(\sum_m |\phi_m(z)|^2 \right)^{1/2} = K_\mu(z, z)^{1/2}. \tag{2.13}$$

Then, if X is a compact subset of D, we have that for any $z \in X$

$$\|E_z\|_\mu \leq C_X := \sup_{w \in X} K_\mu(w, w)^{1/2}.$$

(ii) $\Rightarrow$ (iii). Let $z \in D$ and let $r > 0$ be such that $B(z, 3r) \subset D$. Set $Y := \overline{B(z, r)}$ and $X := \overline{B(z, 2r)}$. If $C_X > 0$ is such a constant that (2.10) holds then for any $z' \in \partial Y$ we can take $V_{z'} := B(z', r)$ and $C_{z'} := C_X$.

(iii) $\Rightarrow$ (iv). Let $z \in D$ and let Y be chosen as in the statement (iii) above. If $z \in \partial Y$ then the existence of the neighborhood V_z and the constant C_z follows immediately form (iii). Suppose now that $z \in \operatorname{int} Y$. Choose for each $z' \in \partial Y$ a neighborhood $V_{z'}$ and a constant $C_{z'}$ such as in (iii). Since ∂Y is compact we can find a finite number of points $z_1, z_2, \ldots, z_k \in \partial Y$ such that

$$\partial Y \subset \bigcup_{i=1}^{k} V_{z_i}.$$

Then for any $z' \in \partial Y$ and any $f \in L^2 H(D, \mu)$

$$| f(z') | = | E_{z'} f | \leq C_z \| f \|_\mu$$

where $C_z := \max \{ C_{z_1}, C_{z_2}, \ldots, C_{z_k} \}$. Using now the maximum principle for holomorphic functions we obtain that (2.12) holds for each $w \in V_z := \operatorname{int} Y$.

(iv) $\Rightarrow$ (i). The continuity of evaluation functionals follows immediately from (2.12). Moreover, if $\{ f_m \}$ is a sequence of elements of $L^2 H(D, \mu)$ which converges in $L^2(D, \mu)$ to a function $f \in L^2(D, \mu)$ then, by (iv), this sequence converges locally uniformly on D. Hence, by the Weierstrass theorem on the limit of a uniformly convergent sequence of holomorphic functions, the function f is holomorphic, i.e., $f \in L^2 H(D, \mu)$. This implies that $L^2 H(D, \mu)$ is a closed subspace of $L^2(D, \mu)$.

REMARK. The notion of a μ-Bergman function is not necessary for the proof of the implication (i) $\Rightarrow$ (ii) in the above theorem. One can prove this implication directly from Definition 2.1 using the Banach-Steinhaus theorem on sequences of linear continuous operators (Rudin [9] 5.8).

3. SUFFICIENT CONDITIONS FOR A WEIGHT TO BE AN ADMISSIBLE WEIGHT.

A verification of conditions (ii), (iii) or (iv) from Theorem 2.2 usually needs additional considerations and it is sometimes a sufficiently difficult task. In this section we give more effective sufficient conditions for a weight to be an admissible weight.

THEOREM 3.1. Assume that $\mu \in W(D)$, U is an open subset of D and there exists a number $a > 0$ such that the function μ^{-a} is integrable on U with respect to the Lebesque measure. Then for any $z \in U$ there exists a neighborhood V_z of z in D and a constant $C_z > 0$ such that for each $w \in V_z$ and each $f \in L^2 H(D, \mu)$

$$| E_w f | \leq C_z \| f \|_\mu$$

PROOF. If $x^0 \in \mathbf{C}^n$, $R > 0$, define the open ball $B(x^0, R) := \{ x \in \mathbf{C}^n : | x - x^0 | < R \}$. Let $z \in U$ and let a number $r > 0$ be such that $B(z, 2r) \subset U$. Put $V_z := B(z, r)$, $p := (1 + a)/a$ and $q := 1 + a$. We have that, for every $w \in V_z$, $\overline{B(w, r)} \subset U$. (The symbol $\overline{X}$ denotes the closure of the set X.) If $f \in L^2 H(D, \mu)$ then $| f |^{2/p}$ is a subharmonic function (see [6]), Corollary 2.1.15 or (Hervé [10] 1.2.2) and therefore

$$| f(w) |^{\frac{2}{p}} \leq \frac{1}{\operatorname{vol} B(w, r)} \int_{B(w, r)} | f(x) |^{\frac{2}{p}} (dx)^{2n}, \quad w \in V_z$$

which is the mean value property, (see [6] Theorem 2.14). Since $p, q > 1$ and $1/p + 1/q = 1$ we have

by the Hölder inequality

$$\operatorname{vol} B(w,r) \, | f(w) |^{\frac{2}{p}} \leq \int\limits_{B(w,r)} | f(x) |^{\frac{2}{p}} \mu(x)^{\frac{1}{p}} \mu(x)^{-\frac{1}{p}} (dx)^{2n}$$

$$\leq \left(\int\limits_{B(w,r)} | f(x) |^2 \mu(x) (dx)^{2n} \right)^{\frac{1}{p}} \left(\int\limits_{B(w,r)} \mu(x)^{-\frac{q}{p}} (dx)^{2n} \right)^{\frac{1}{q}}$$

$$\leq \| f \|_{\mu}^{2/p} \left(\int\limits_{U} \mu(x)^{-a} (dx)^{2n} \right)^{\frac{1}{q}}, \quad w \in V_z.$$

Then

$$| f(w) | \leq \left(\int\limits_{U} \mu(x)^{-a} (dx)^{2n} \right)^{\frac{p}{2q}} \left(\operatorname{vol} B(w,r) \right)^{-\frac{p}{2}} \| f \|_{\mu}$$

and we can take

$$C_z := \left(\int\limits_{U} \mu(x)^{-a} (dx)^{2n} \right)^{\frac{1}{2a}} \left(\operatorname{vol} B(w,r) \right)^{-\frac{1+a}{2a}}.$$

COROLLARY 3.1. Let $\mu \in W(D)$. Assume that for each $z \in D$ there exists a compact set $Y \subset D$ which contains z and has the following property: for any $w \in \partial Y$ there exists a neighborhood U_w of w in D and a number $a_w > 0$ such that the function μ^{-a_w} is integrable on U_w with respect to the Lebesque measure. Then $\mu \in AW(D)$. If in particular, the function μ^{-a} is locally integrable on D for some $a > 0$ then $\mu \in AW(D)$.

PROOF. See Theorems 2.2 and 3.1.

COROLLARY 3.2. Let $\mu \in W(D)$. If for any $z \in D$ there exist a neighborhood U of z in D, a constant $a > 0$ and a function $f \in C^1(U, R)$ such that:

(i) 0 is a regular value of f;

(ii) for almost all $w \in U$

$$\mu(w) \geq | f(w) |^a,$$

then $\mu \in AW(D)$.

PROOF. Let $z \in D$ and let U, a and f be associated with z as in the assumptions of the Corollary. If $f(z) \neq 0$ then there exists a bounded neighborhood U' of z in D and a constant $b > 0$ such that for any $w \in U'$

$$| f(w) |^a > b.$$

Hence $\mu(w)^{-1} \leq | f(w) |^{-a} < b^{-1}$, i.e., μ^{-1} is an integrable function on U'.

Suppose now that $f(z) = 0$. The function f is regular in z which implies that there exists $r > 0$ and a bounded diffeomorphism $F : B(0,r) \to U$ such that $F(0) = z$ and for each $w = (w_1, ..., w_n) \in B(0,r)$

$$(f \circ F)(w) = \operatorname{Im} w_n.$$

Since $| \operatorname{Im} w_n |^{-\frac{1}{2}}$ is an integrable function on $B(0,r)$ we obtain that $| f |^{-\frac{1}{2}}$ is integrable on $U' := F(B(0,r))$. Hence

$$\mu^{-\frac{1}{2a}} \leq | f |^{-\frac{1}{2}}$$

is an integrable function on U'. The theorem follows now from Corollary 3.1.

EXAMPLE 3.1. Let D be the unit disk in $\mathbf{C}$. Let for any $z \in D$

$$f_t(z) := |\operatorname{Im} z|^t, \quad t \in (0, +\infty)$$

$$g(z) := |\operatorname{Im} z|^{\frac{1}{1-|z|}};$$

$$h(z) := \begin{cases} e^{|z|^{-\frac{1}{2}}} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

Then the functions g, h and f_t, $t \in (0, +\infty)$ are a-weights on D.

4. A WEIGHT WHICH IS NOT AN ADMISSIBLE WEIGHT.

In the sequel we shall use the following theorem.

THEOREM 4.1. (Runge). Let X be a compact subset of $\mathbf{C}$ whose complement is connected. Let $f: X \to \mathbf{C}$ be continuous on X and holomorphic on the interior of X. Then f is the uniform limit on X of holomorphic polynomials. (See [9], 13.9). Now we are in position to give the example of the weight which is not an a-weight. Let $D := \{z \in \mathbf{C}: |z| < 1\}$ be the unit disk in $\mathbf{C}$. Let $n \in \mathbb{N}$. Define

$$A_n := B(0, 2^{-n}) \cup \{z \in D: \operatorname{Re} z > 0, |\operatorname{Im} z| < 2^{-n}\},$$

$$M_n := (\overline{D} - A_n) \cup \overline{A}_{n+1}$$

where $\overline{D}$ and $\overline{A}_{n+1}$ denotes the closure in $\mathbf{C}$ of D and A_{n+1} respectively. Let $f_n: M_n \to \mathbf{C}$ be given the formula

$$f_n(z) := \begin{cases} 1 + \frac{1}{n} & \text{for } z \in \overline{A}_{n+1} \\ 0 & \text{for } z \in \overline{D} - A_n. \end{cases}$$

Since the set $\mathbf{C} - M_n$ is connected we obtain by the Runge theorem that there exists a holomorphic polynomials g_n such that for any $z \in M_n$

$$|f_n(z) - g_n(z)| < 1/n.$$

We have that $|g_n(z)| < 1/n$ for $z \in \overline{D} - A_n$ and $1 < |g_n(z)| < 1 + 2/n$ for $z \in \overline{A}_{n+1}$. Let

$$h_n(z) := \frac{1}{g_n(0)} g_n(z), \quad z \in D.$$

Since $|g_n(0)| > 1$ we have that $|h_n(z)| < 1/n$ for $z \in D - A_n$ and $|h_n(z)| < 1 + 2/n$ for $z \in \overline{A}_{n+1} \cap D$. Moreover, $h_n(0) = 1$.

Now define a weight $\mu \in W(D)$ as follows:

$$\mu(z) := \begin{cases} 1 & \text{for } z \in D - A_1 \\ 0 & \text{for } z \in [0,1) \subset R \\ \min_{n \in \mathbb{N}} \{1, 1/|h_n(z)|^2\} & \text{for } z \in A_n - A_{n+1}. \end{cases}$$

We have that for any $z \in D$, $\mu(z) \leq 1$. It is clear that $h_n \in L^2 H(D, \mu)$ for $n \in \mathbb{N}$ and $E_0 h_n = 1$.

Notice that for each $z \in D$

$$|h_n(z)|^2 \mu(z) < 9 \quad \text{and} \quad \lim_{n \to \infty} |h_n(z)|^2 \mu(z) = 0.$$

Then by the Lebesque theorem on majorized convergence

$$\lim_{n \to \infty} \|h_n\|_\mu^2 = \lim_{n \to \infty} \int_D |h_n(z)|^2 \mu(z)(dz)^{2n} = 0,$$

whereas $\lim_{n \to \infty} E_0 h_n = 1$. This implies that the evaluation functional E_0 is not continuous on $L^2 H(D, \mu)$ and therefore μ is not an admissible weight on D.

5.   THE WEIGHT $\mu(z) = (\text{Im } z)^2$ ON THE UNIT DISC IN $\mathbb{C}$.

We are going to find a complete orthonormal system in $L^2 H(D, \mu)$. Let for any $n \in \mathbb{Z}^+$

$$g_n(z) := z^n, \quad z \in D.$$

If $r > 0$ define $Dr := \{z \in \mathbb{C} : |z| < r\}$. Since

$$(\text{Im } z)^2 = \tfrac{1}{2}|z|^2 - \tfrac{1}{4}(z^2 + \overline{z}^2)$$

one can easy verify that

$$\int_{Dr} \overline{g_m(z)}\, g_k(z)\, \mu(z)\, d^2 z := \begin{cases} \dfrac{\pi r^{2m+4}}{2(m+2)} & \text{for } k = m, \\[2mm] -\dfrac{\pi r^{2m+2}}{4(m+1)} & \text{for } k = m-2, \\[2mm] -\dfrac{\pi r^{2m+6}}{4(m+3)} & \text{for } k = m+2, \\[2mm] 0 \ \text{ in the other cases.} \end{cases} \tag{5.1}$$

Setting $r = 1$ we obtain

$$<g_m \mid g_k>_\mu := \begin{cases} \dfrac{\pi}{2(m+2)} & \text{for } k = m, \\[2mm] -\dfrac{\pi}{4(m+1)} & \text{for } k = m-2, \\[2mm] -\dfrac{\pi}{4(m+3)} & \text{for } k = m+2, \\[2mm] 0 \ \text{ in the other cases.} \end{cases} \tag{5.2}$$

Notice that for any $k, l \in \mathbb{Z}^+$

$$<g_{2k} \mid g_{2l+1}>_\mu = 0.$$

This implies that there exists such an orthogonal system $(f_n)_{n=0}^\infty$ in $L^2 H(D, \mu)$ that for any $j \in \mathbb{Z}^+$

$$\begin{cases} f_{2j} \in \text{Span}\{g_0,\ g_2, ..., g_{2j}\}, \\[2mm] f_{2j+1} \in \text{Span}\{g_1, g_3, ..., g_{2j+1}\}. \end{cases} \tag{5.3}$$

(In particular the sequence obtained from (g_n) by the Schmidts orthonormalization procedure has the above property.) Let (f_n) be such a system. We have

$$
\begin{cases}
f_{2j} = \sum_{k=0}^{j} a_{2j,2k}\, g_{2k}, \\[2ex]
f_{2j+1} = \sum_{k=0}^{j} a_{2j+1,\,2k+1} g_{2k+1},
\end{cases}
\qquad j \in \mathbf{Z}^+
\tag{5.4}
$$

and

$$
<f_n \,|\, g_m>_\mu = 0
\tag{5.5}
$$

for each $0 \le m < n$, $n \in \mathbf{N}$. By (5.2), (5.4) and (5.5) we get

$$
0 = \; <g_m \,|\, \sum_{p=0}^{n} a_{n,p}\, g_p>_\mu = -\frac{\pi}{4(m+1)}\, a_{n,m-2} + \frac{\pi}{2(m+2)}\, a_{n,m} - \frac{\pi}{4(m+3)}\, a_{n,m+2}
$$

for $0 \le m < n$. Then we have the following recurrent formulas

$$
\begin{cases}
a_{2j,2} = 3a_{2j,0},\, a_{2j+1,3} = \dfrac{8}{3} a_{2j+1,1}, \qquad j \in \mathbf{Z}^+ \\[2ex]
a_{n,m+2} = 2\,\dfrac{m+3}{m+2}\, a_{n,m} - \dfrac{m+3}{m+1}\, a_{n,m-2} \text{ for } m > 2, \quad n \in \mathbf{Z}^+
\end{cases}
\tag{5.6}
$$

where $a_{2j,0}$ and $a_{2j+1,1}$ are arbitrary complex numbers. Notice that, by (5.4), $a_{n,m} = 0$ for $n - m$ to be an odd number. By (5.6) it is clear that $a_{2j,2k}$ is a linear function of the variable $a_{2j,0}$, whereas $a_{2j+1,2k+1}$ is a linear function of $a_{2j+1,1}$ for $k \le j$.

From now on we fix (f_n) taking

$$
a_{2j,0} = a_{2j+1,1} = 1
$$

for each $j \in \mathbf{Z}^+$. If $i, j, k \in \mathbf{Z}^+$ and $k \le \min\{i,j\}$ then

$$
a_{2j,2k} = a_{2i,2k} \text{ and } a_{2j+1,2k+1} = a_{2i+1,2k+1}.
$$

Therefore we shall write a_k instead of $a_{n,k}$ for each $n, k \in \mathbf{Z}^+$, $k \le n$. We have

$$
\begin{aligned}
f_{2j} &= \sum_{k=0}^{j} a_{2k}\, g_{2k} \\[2ex]
f_{2j+1} &= \sum_{k=0}^{j} a_{2k+1}\, g_{2k+1}, \quad j \in \mathbf{Z}^+,
\end{aligned}
\tag{5.7}
$$

where

$$
a_0 = 1,\, a_1 = 1,\; a_2 = 3,\; a_3 = \frac{3}{8},
$$

$$
a_{m+2} = 2\frac{m+3}{m+2}\, a_m - \frac{m+3}{m+1}\, a_{m-2} \text{ for } m \ge 2.
\tag{5.8}
$$

It is obvious that all a_k are real numbers.

Let us calculate the norm $\| f \|_\mu$. By (5.2) and (5.8) we get

$$\| f_0 \|^2_\mu = \frac{\pi}{4}, \quad \| f_1 \|^2_\mu = \frac{\pi}{6}$$

$$\| f_m \|^2_\mu = <f_m \,|\, f_{m-2} + a_m g_m>_\mu = a_m <f_m \,|\, g_m>_\mu =$$

$$= a_m\left(-\frac{\pi}{4(m+1)}\, a_{m-2} + \frac{\pi}{2(m+2)}\, a_m \right)$$

$$= \frac{\pi}{4(m+3)}\, a_m a_{m+2}, \qquad \text{for } m \geq 2. \tag{5.9}$$

Since $f_m \neq 0$ we obtain that $a_m \neq 0$ for each $m \in \mathbb{Z}^+$. Now define a sequence (ϕ_m) of elements of $L^2 H(D,\mu)$ by the formula:

$$\phi_m := \frac{1}{\| f_m \|_\mu}\, f_m = 2\left(\frac{m+3}{\pi a_m a_{m+2}}\right)^{\frac{1}{2}} f_m. \tag{5.10}$$

We are going to show that the orthonormal system (ϕ_m) is complete in $L^2 H(D,\mu)$.

LEMMA 5.1. If (a_m) is a sequence defined by (5.8) then for any $m \in \mathbb{N}$

$$\left(1 + \frac{3}{m+2}\right) a_m < a_{m+2} < \left(1 + \frac{4}{m}\right) a_m. \tag{5.11}$$

PROOF. By a direct calculation we can verify (5.11) for $m = 1$ and $m = 2$. Suppose now that it is true for each $m \leq n-1$, where $n \in \mathbb{N}$ and $n > 1$. In particular

$$\left(1 + \frac{3}{n}\right) a_{n-2} < a_n = a_{(n-2)+2} < \left(1 + \frac{4}{n-2}\right) a_{n-2}. \tag{5.12}$$

Then

$$a_{n-2} < \frac{n}{n+3} a_n$$

and

$$a_{n+2} = \frac{2n+6}{n+2}\, a_n - \frac{n+3}{n+1} a_{n-2} > \left(\frac{2n+6}{n+2} - \frac{n}{n+1}\right) a_n = \frac{n^2 + 6n + 6}{(n+1)(n+2)} a_n.$$

Since

$$\frac{n^2 + 6n + 6}{(n+1)(n+2)} > \frac{n+5}{n+2}$$

we get

$$a_{n+2} > \frac{n+5}{n+2} a_n = \left(1 + \frac{3}{n+2}\right) a_n.$$

From (5.12) it follows that

$$a_{n-2} > \frac{n-2}{n+2} a_n.$$

Hence

$$a_{n+2} = \frac{2n+6}{n+2}\, a_n - \frac{n+3}{n+1} a_{n-2}$$

$$< \left(\frac{2n+6}{n+2} - \frac{n+3}{n+1}\frac{n-2}{n+2}\right) a_n = \frac{n^2 + 7n + 12}{(n+1)(n+2)} a_n.$$

Since

$$\frac{n+4}{n} - \frac{n^2 + 7n + 12}{(n+1)(n+2)} = \frac{2n+8}{n(n+1)(n+2)} > 0$$

we obtain

$$a_{n+2} < \frac{n+4}{n} a_n = (1 + \tfrac{4}{n}) a_n.$$

Finally we can state that (5.11) is true for $m = n$ and by the induction principle it is true for any $m \in \mathbb{N}$.

COROLLARY 5.1. The series $\sum\limits_{n=0}^{\infty} a_{2j} z^{2j}$ and $\sum\limits_{j=0}^{\infty} a_{2j+1} z^{2j+1}$ converge on D.

PROOF. By (5.11) we get $\lim\limits_{j \to \infty} \frac{a_{2j+2}}{a_{2j}} = \lim\limits_{j \to \infty} \frac{a_{2j+3}}{a_{2j+1}} = 1$.

Hence, by d'Alembert's test, the series

$$\sum\limits_{j=0}^{\infty} a_{2j} w^j \text{ and } \sum\limits_{j=0}^{\infty} a_{2j+1} w^j \text{ converge for } |w| < 1.$$

Taking $w = z^2$ we obtain that the series $\sum\limits_{j=0}^{\infty} a_{2j} z^{2j}$ and $\sum\limits_{j=0}^{\infty} a_{2j+1} z^{2j+1} = z \cdot \sum\limits_{j=0}^{\infty} a_{2j+1} z^{2j}$ converge on D.

LEMMA 5.2. Let (a_n) be a sequence defined by the formula (5.8). Then for any $j \in \mathbb{N}$

$$a_{2j}^2 > \frac{(2j+3)(j+2)(j+1)}{6}; \qquad (5.13)$$

$$a_{2j+1}^2 > \frac{(2j+5)(2j+3)(j+2)}{30}. \qquad (5.14)$$

PROOF. Using j times the inequality (5.11) we have

$$a_{2j} > \prod\limits_{k=1}^{j} (1 + \tfrac{3}{2k}), \; a_{2j+1} > \prod\limits_{k=1}^{j} (1 + \tfrac{3}{2k+1}). \qquad (5.15)$$

Let

$$A_{2j} := \prod\limits_{k=1}^{j} (1 + \tfrac{3}{2k}), \; A_{2j+1} := \prod\limits_{k=1}^{j} (1 + \tfrac{3}{2k+1}).$$

It is clear that $A_{2j} > A_{2j+1}$. Hence

$$A_{2j}^2 > A_{2j} A_{2j+1} = \frac{2j+4}{2j+1} \frac{2j+3}{2j} \frac{2j+2}{2j-1} \frac{2j+1}{2j-2} \cdots \frac{8}{5} \frac{7}{4} \frac{6}{3} \frac{5}{2}$$

$$= \frac{(2j+4)(2j+3)(2j+2)}{4 \cdot 3 \cdot 2}$$

which implies (5.13). We have also

$$A_{2j+2} = \frac{5}{2} \prod\limits_{k=2}^{j+1} (1 + \tfrac{3}{2k}) = \frac{5}{2} \prod\limits_{k=1}^{j} (1 + \tfrac{3}{2k+2}) < \frac{5}{2} A_{2j+1}.$$

Then

$$\frac{5}{2} A_{2j+1}^2 > A_{2j+2} A_{2j+1} = \frac{2j+5}{2j+2} \frac{2j+4}{2j+1} \frac{2j+3}{2j} \frac{2j+2}{2j-1} \cdots \frac{8}{5} \frac{7}{4} \frac{6}{3} \frac{5}{2}$$

$$= \frac{(2j+5)(2j+4)(2j+3)}{4 \cdot 3 \cdot 2}$$

which implies (5.14).

THEOREM 5.1. The system (ϕ_m) given by the formula (5.10) is complete in $L^2 H(D, \mu)$.

PROOF. Let $f(z) = \sum\limits_{k=0}^{\infty} b_k z^k$ be an element of $L^2 H(D, \mu)$ which is orthogonal to any ϕ_m,

$m \in \mathbf{Z}^+$. Since for each $m \in \mathbf{Z}^+ g_m \in \mathrm{Span}\{\phi_1 \ \phi_2, ...\}$ we have $<f \,|\, g_m>_\mu = 0$. On the other hand, for any $0 < r < 1$ the series $\sum\limits_{k=0}^{\infty} b_k z^k$ converges to $f(z)$ uniformly on the disk Dr.

Therefore, by (5.1)

$$\int\limits_{Dr} \overline{g_0(z)}\, f(z)\mu(z)d^2z = \sum\limits_{k=0}^{\infty} b_k \int\limits_{Dr} z^k \mu(z)\, d^2z = \frac{\pi r^4}{4}\, b_0 - \frac{\pi r^6}{12}\, b_2.$$

Similarly

$$\int\limits_{Dr} \overline{g_1(z)}\, f(z)\mu(z)d^2z = \frac{\pi r^6}{6}\, b_1 - \frac{\pi r^8}{16}\, b_3$$

and

$$\int\limits_{Dr} \overline{g_m(z)} f(z)\mu(z)d^2z = -\frac{\pi r^{2m+2}}{4(m+1)}\, b_{m-2} + \frac{\pi r^{2m+4}}{2(m+2)} - \frac{\pi r^{2m+6}}{4(m+3)}, \quad \text{for } m > 1.$$

If $r \to 1$ we obtain equations which are identical with the equations (5.6) in which $a_{n,m}$ are exchanged for b_m, $m \in \mathbf{Z}^+$. This implies that for any $j \in \mathbf{Z}^+$

$$b_{2j} = a_{2j}\, b_0 \quad \text{and} \quad b_{2j+1} = a_{2j+1} b_1, \quad b_0, b_1 \in \mathbf{C}$$

where (a_m) is the sequence defined by (5.8).

Let $F_0(z) = \sum\limits_{j=0}^{\infty} a_{2j}z^{2j}$ and $F_1(z) = \sum\limits_{j=0}^{\infty} a_{2j+1}z^{2j+1}, z \in D$. From Corollary 5.1 it follows that F_1 and F_2 are well defined holomorphic functions on D. We have $f = b_0 F_0 + b_1 F_1$. Since by (5.1)

$$\int\limits_{Dr} \overline{F_0(z)}\, F_1(z)\mu(z)d^2z = \sum\limits_{j,k=0}^{\infty} a_{2j}a_{2k+1} \int\limits_{Dr} \overline{g_{2j}(z)}g_{2k+1}(z)\mu(z)d^2z = 0$$

we obtain that

$$\int\limits_{Dr} |\,f(z)\,|^2 \mu(z)d^2z = |\,b_0\,|^2 \int\limits_{Dr} |\,F_0(z)\,|^2 \mu(z)d^2z + |\,b_1\,|^2 \int\limits_{Dr} |\,F_1(z)\,|^2 \mu(z)d^2z.$$

Hence

$$\|\,f\,\|_\mu^2 = \lim\limits_{r\to 1} \int\limits_{Dr} |\,f(z)\,|^2 d^2z = \lim\limits_{r\to 1} \left[|\,b_0\,|^2 \cdot \int\limits_{Dr} |\,F_0(z)\,|^2 \mu(z)d^2z + |\,b_1\,|^2 \int\limits_{Dr} |\,F_1(z)\,|^2 \mu(z)d^2z \right]$$

Notice that

$$\int\limits_{Dr} |\sum\limits_{k=0}^{n} a_{2k}z^{2k}|^2 \mu(z)d^2z = \sum\limits_{k,m=0}^{n} a_{2k}a_{2m} \int\limits_{Dr} \overline{g_{2k}(z)}\, g_{2m}(z)\mu(z)d^2z$$

$$= \sum\limits_{m=0}^{n} \frac{\pi r^{4m+4}}{2(2m+2)}\, a_{2m}^2 - \sum\limits_{m=1}^{n} \frac{\pi r^{4m+2}}{4(2m+1)}\, a_{2m-2}\, a_{2m} - \sum\limits_{m=0}^{n-1} \frac{\pi r^{4m+6}}{4(2m+3)}\, a_{2m}\, a_{2m+2}.$$

After further calculations we get

$$\int_{Dr} \Big| \sum_{k=0}^{n} a_{2k} z^{2k} \Big|^2 \mu(z) d^2 z = \frac{\pi r^{4n+4}}{4} \frac{a_{2n}}{2n+3} a_{2n+2} + \frac{\pi(1-r^2)^2}{4} \sum_{k=0}^{n-1} r^{4k+4} \frac{a_{2k}}{2k+3} a_{2k+2}.$$

Using Lemma 5.1 and 5.2 we see that

$$\lim_{n \to \infty} \frac{\pi}{4} r^{4n+4} \frac{a_{2n}}{2n+3} a_{2n+2} = 0$$

and

$$\int_{Dr} |F_0(z)|^2 \mu(z) d^2 z = \frac{\pi(1-r^2)^2}{4} \sum_{k=0}^{\infty} r^{4k+4} \frac{a_{2k}}{2k+3} a_{2k+2}$$

$$> \frac{\pi(1-r^2)^2}{4} \cdot \sum_{k=0}^{\infty} r^{4k+4} a_{2k}^2 \frac{2k+5}{(2k+2)(2k+3)}$$

$$> \frac{5\pi r^4 (1-r^2)^2}{24} + \frac{\pi(1-r^2)^2}{4} \sum_{k=1}^{\infty} \frac{(2k+5)(2k+3)(k+2)(k+1)}{12(k+1)(2k+3)} r^{4k+4}$$

$$= \frac{\pi(1-r^2)^2}{48} \sum_{k=0}^{\infty} (2k+5)(k+2) r^{4k+4}$$

Let $s := r^4$. Since $\frac{2k+5}{k+3} > \frac{5}{3}$ we have

$$\sum_{k=0}^{\infty} (2k+5)(k+2) r^{4k+4} > \frac{5}{3} \sum_{k=0}^{\infty} (k+3)(k+2) s^{k+1} = \frac{5}{3} \frac{2s^3 - 6s^2 + 6s}{(1-s)^3} = \frac{5}{3} r^4 \frac{2r^8 - 6r^4 + 6}{(1-r^2)^3 (1+r^2)^3}.$$

Hence

$$\int_{Dr} |F_0(z)|^2 \mu(z) d^2 z > \frac{5\pi}{72} r^4 \frac{r^8 - 3r^4 + 3}{(1-r^2)(1+r^2)^3}$$

and as a consequence

$$\lim_{\substack{r \to 1 \\ r < 1}} \int_{Dr} |F_0(z)|^2 \mu(z) \, d^2 z = +\infty. \tag{5.18}$$

Analogously, one can show that for any $0 < r < 1$

$$\int_{Dr} |F_1(z)|^2 \mu(z) d^2 z > \frac{\pi}{30} \frac{r^{12} - 6r^8 + 6r^4}{(1-r^2)(1+r^2)^3}$$

and therefore

$$\lim_{\substack{r \to 1 \\ r < 1}} \int_{Dr} |F_1(z)|^2 \mu(z) d^2 z = +\infty. \tag{5.19}$$

Formulas (5.17), (5.18), and (5.19) imply that $b_0 = b_1 = 0$ and as a consequence $f = 0$. This completes the proof of our theorem.

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