ON θ -C-OPEN SETS

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ABSTRACT. The properties of the collection of complements of θ -closures of sets in a topological space are investigated in this paper. A strong continuity condition is defined in terms of these sets. Some applications to H-closed spaces and Katetov spaces are given.

KEY WORDS AND PHRASES. θ -C-open, θ -C-continuous, super-continuous, H-closed space, Katetov space.
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1. PRELIMINARIES.

All spaces are topological spaces with no separation axioms assumed unless explicitly stated. Let A be a subset of a space X . The closure of A and the interior of A are denoted by $Cl A$ and $Int A$, respectively. The set A is said to be regular open (regular closed) if $A = Int Cl A$ ($A = Cl Int A$). The θ -closure ($\mathcal{S}$ -closure) (Velicko [1]) of A is the set of all x in X such that every closed neighborhood (the interior of every closed neighborhood) of x intersects A nontrivially. The θ -closure and the $\mathcal{S}$ -closure of A are denoted by $Cl_\theta A$ and $Cl_\mathcal{S} A$, respectively. The set A is called θ -closed ($\mathcal{S}$ -closed) if $A = Cl_\theta A$ ($A = Cl_\mathcal{S} A$). A set A is said to be θ -open ($\mathcal{S}$ -open) if its complement is θ -closed ($\mathcal{S}$ -closed). For a given space X both the collection of all θ -open sets and the collection of all $\mathcal{S}$ -open sets form topologies. The collection of $\mathcal{S}$ -open sets is usually referred to as the semi-regular topology.

DEFINITION 1. Arya and Gupta [2]. A function $f: X \rightarrow Y$ is said to be completely continuous if for each open subset V of Y , $f^{-1}(V)$ is regular open in X .

DEFINITION 2. Munshi and Basson [3]. A function $f: X \rightarrow Y$ is said to be super-continuous if for each $x \in X$ and each open

neighborhood V of $f(x)$, there exists an open neighborhood U of x for which $f(\text{Int } Cl U) \subseteq V$.

DEFINITION 3. Long and Herrington [4]. A function $f: X \rightarrow Y$ is said to be strongly θ -continuous if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x for which $f(Cl U) \subseteq V$.

DEFINITION 4. Porter and Tikoo [5]. A space X is said to be H -closed if X is a closed subset in every space containing X as a subspace.

DEFINITION 5. Porter and Tikoo [5]. A space is said to be Katetov if it has a coarser minimal H -closed topology or equivalently a coarser H -closed topology.

2. θ -C-OPEN SETS

We define a subset U of a space X be θ -C-open provided there exists a subset A of X for which $X - U = Cl_{\theta} A$. We call a set θ -C-closed if its complement is θ -C-open or equivalently if there is a subset A of X such that the set equals $Cl_{\theta} A$.

THEOREM 1. If U is θ -open, then U is θ -C-open.

PROOF. Since U is θ -open, $X - U$ is θ -closed. Hence $X - U = Cl_{\theta}(X - U)$.

THEOREM 2. If U is open, then $\text{Int } Cl U$ is θ -C-open.

PROOF. $\text{Int } Cl U = X - Cl(X - Cl U)$. Since $X - Cl U$ is open, $Cl(X - Cl U) = Cl_{\theta}(X - Cl U)$ (Velicko [1]). Hence $X - \text{Int } Cl U = Cl_{\theta}(X - Cl U)$.

COROLLARY. If U is regular open, then U is θ -C-open.

Since the real numbers with the usual topology contain θ -open sets that are not regular open, it follows that the real numbers contain θ -C-open sets that are not regular open.

THEOREM 3. Regular openness is equivalent to θ -C openness if and only if $Cl_{\theta} A$ is regular closed for every set A .

PROOF. Let X be a space. Assume regular openness is equivalent to θ -C-openness and let $A \subseteq X$. Then $X - Cl_{\theta} A$ is regular open. Thus $X - Cl_{\theta} A = \text{Int } Cl(X - Cl_{\theta} A) = \text{Int}(X - \text{Int } Cl_{\theta} A) = X - Cl \text{Int } Cl_{\theta} A$. Therefore $Cl_{\theta} A = Cl \text{Int } Cl_{\theta} A$ which implies that $Cl_{\theta} A$ is regular closed.

Assume $Cl_{\theta} A$ is regular closed for every set A . Suppose U is θ -C-open and let $A \subseteq X$ such that $U = X - Cl_{\theta} A$. Then $\text{Int } Cl U = \text{Int } Cl(X - Cl_{\theta} A) = \text{Int}(X - \text{Int } Cl_{\theta} A) = X - Cl \text{Int } Cl_{\theta} A = X - Cl_{\theta} A = U$. Therefore U is regular open and hence regular openness is equivalent to θ -C-openness.

THEOREM 4. If U is θ -C-open, then U is a union of regular open sets (that is ε -open).

PROOF. Let U be θ -C-open. Let $x \in U$. Since U is θ -C-open, there exists a set $A \subseteq X$ such that $U = X - Cl_{\theta} A$. Because $x \notin Cl_{\theta} A$, there exists an open set W for which $x \in W$ and $(Cl W) \cap A = \emptyset$. Hence $x \in \text{Int } Cl W \subseteq X - Cl_{\theta} A = U$. Thus U is a union of regular open sets.

It follows from Theorem 4 and the corollary to Theorem 2 that the θ -C-open sets form a basis for the semi-regular topology.

THEOREM 5. The intersection of two θ -C-open sets is θ -C-open.

PROOF. Let U and V be θ -C-open sets. There exist sets A and B such that $U = X - Cl_{\theta} A$ and $V = X - Cl_{\theta} B$. Then $U \cap V = (X - Cl_{\theta} A) \cap (X - Cl_{\theta} B) = X - (Cl_{\theta}(A) \cup Cl_{\theta}(B)) = X - Cl_{\theta}(A \cup B)$.

The following example shows that the union of two θ -C-open sets need not be θ -C-open. It follows that the θ -C-open sets do not form a topology and hence θ -C-openness is not equivalent to either δ -openness or θ -openness.

EXAMPLE 1. Let $X = \{a, b, c\}$ and $\mathcal{O} = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. The θ -C-open sets of X are $X, \emptyset, \{a\}$, and $\{c\}$.

3. θ -C-CONTINUITY.

We define a function $f: X \rightarrow Y$ to be θ -C-continuous if for each open subset V of Y , $f^{-1}(V)$ is θ -C-open in X . Since θ -C-open sets are open, obviously θ -C-continuity implies continuity. Since by Theorem 2 regular openness implies θ -C-openness, complete continuity implies θ -C-continuity. The identity mapping on the real numbers with the usual topology is θ -C-continuous but not completely continuous.

THEOREM 6. (Munshi and Basson [3]) A function $f: X \rightarrow Y$ is super-continuous if and only if the inverse image of each open set in Y is δ -open in X .

By Theorem 4 every θ -C-open set is δ -open. Hence θ -C-continuity implies super-continuity. The identity mapping on the space in Example 1 is super-continuous but not θ -C-continuous.

It also follows from Theorem 4 that the corresponding "local" or "pointwise" version of θ -C-continuity is equivalent to super-continuity.

THEOREM 7. A function $f: X \rightarrow Y$ is super-continuous if and only if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists a θ -C-open set $U \subseteq X$ for which $x \in U$ and $f(U) \subseteq V$.

THEOREM 8. (Long and Herrington [4]). A function $f: X \rightarrow Y$ is strongly θ -continuous if and only if the inverse image of each open set in Y is θ -open in X .

From Theorem 1 θ -openness implies θ -C-openness. Hence strong θ -continuity implies θ -C-continuity.

EXAMPLE 2. Let $X = \{a, b, c\}$, $\mathcal{O}_1 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\mathcal{O}_2 = \{X, \emptyset, \{a\}\}$. The identity mapping $(X, \mathcal{O}_1) \rightarrow (X, \mathcal{O}_2)$ is θ -C-continuous but not strongly θ -continuous.

Based upon the above theorems and remarks, we have the following implications, none of which are reversible.

complete continuity }
strong θ -continuity } $\Rightarrow \theta$ -C-continuity $\Rightarrow$ super-continuity

THEOREM 9. If $f: X \rightarrow Y$ is θ -C-continuous and $Cl_{\theta} A$ is regular closed for every subset A of X , then f is completely continuous.

PROOF. By Theorem 3 θ -C-openness is equivalent to regular openness.

THEOREM 10. If $f: X \rightarrow Y$ is θ -C-continuous and for every subset A of X , $\text{Cl}_\theta A$ is θ -closed, then f is strongly θ -continuous.

PROOF. Follows from Theorem 8.

The following theorems and examples illustrate some of the basic properties of θ -C-continuous functions.

THEOREM 11. If $f: X \rightarrow Y$ is θ -C-continuous and $g: Y \rightarrow Z$ is continuous, then $g \circ f: X \rightarrow Z$ is θ -C-continuous.

The proof is routine.

COROLLARY. The composition of two θ -C-continuous functions is θ -C-continuous.

THEOREM 12. Let $f_\alpha: X \rightarrow Y_\alpha$ be a function for each α in A and let $f: X \rightarrow \prod Y_\alpha$ be given by $f(x) = (f_\alpha(x))$. If f is θ -C-continuous, then f_α is θ -C-continuous for each α in A .

PROOF. For each $\alpha \in A$ denote the projection onto Y_α by p_α . Then $f_\alpha = p_\alpha \circ f$ is θ -C-continuous by Theorem 11.

The proof of the next theorem follows from Theorem 12.

THEOREM 13. Let $f: X \rightarrow Y$ be a function and let $g: X \rightarrow X \times Y$ given by $g(x) = (x, f(x))$ be its graph function. If g is θ -C-continuous, then f is θ -C-continuous.

The following example shows that the converse of Theorem 13 does not hold.

EXAMPLE 3. Let $X = \{a, b, c\}$ and $\mathcal{Y} = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Define $f: X \rightarrow X$ by $f(a) = f(b) = f(c) = a$. Then f is θ -C-continuous, but its graph function is not θ -C-continuous since $g^{-1}(\{(a, a), (c, a)\}) = \{a, c\}$ which is not θ -C-open.

The proof of the following theorem is straightforward and is omitted.

THEOREM 14. A function $f: X \rightarrow Y$ is θ -C-continuous if and only if for each closed subset F of Y , there exists a subset A of X for which $f^{-1}(F) = \text{Cl}_\theta A$.

THEOREM 15. If the functions $f, g: X \rightarrow Y$ are θ -C-continuous and Y is Hausdorff, then the set $A = \{x : f(x) \neq g(x)\}$ is a union of θ -C-open sets.

PROOF. Let $x \in A$. Since $f(x) \neq g(x)$ and Y is Hausdorff, there exist disjoint open sets V and W containing $f(x)$ and $g(x)$, respectively. Then $f^{-1}(V)$ and $g^{-1}(W)$ are θ -C-open. By Theorem 5 $f^{-1}(V) \cap g^{-1}(W)$ is θ -C-open. Obviously $x \in f^{-1}(V) \cap g^{-1}(W) \subseteq A$.

COROLLARY. If the functions $f, g: X \rightarrow Y$ are θ -C-continuous, then the set $B = \{x : f(x) = g(x)\}$ is δ -closed.

PROOF. By Theorem 15 $X - B$ is a union of θ -C-open sets and by Theorem 4 each θ -C-open set is a union of regular open sets.

For a function $f: X \rightarrow Y$ the graph of f , denoted by $G(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

THEOREM 16. If $f: X \rightarrow Y$ is θ -C-continuous and Y is Hausdorff, then $X \times Y - G(f)$ is a union of sets of the form $A \times B$ where A is θ -C-open and B is open.

PROOF. Let $(x, y) \in X \cdot Y - G(f)$. There exist disjoint open sets V and W for which $f(x) \in V$ and $y \in W$. Then $f^{-1}(V)$ is θ -C-open and $(x, y) \in f^{-1}(V) \times W \subseteq X \cdot Y - G(f)$.

The following theorem is easily proved.

THEOREM 17. If $f: X \rightarrow Y$ is a θ -C-continuous injection and Y is Hausdorff, then points in X can be separated by θ -C-open sets.

4. APPLICATIONS TO H-CLOSED SPACES AND KATETOV SPACES

In this section all spaces are assumed to be Hausdorff.

Since H-closed spaces and Katetov spaces are related to the θ -closures of sets, there are natural relationships between these spaces and θ -C-open sets and θ -C-continuity. The following results are required.

THEOREM 18. (Porter and Tikoo [5]). If X is an H-closed space and $A \subseteq X$, then $\text{Cl}_{\theta} A$ is Katetov.

THEOREM 19. (Porter and Tikoo [5]). An H-closed space in which every closed set is the θ -closure of some set is compact.

The next result follows immediately from Theorems 14 and 18.

Theorem 20. If X is H-closed and $f: X \rightarrow Y$ is θ -C-continuous, then for each closed subset F of Y , $f^{-1}(F)$ is a Katetov subspace of X .

As a consequence of Theorem 19 we have the following result.

Theorem 21. If X is H-closed and every open set is θ -C-open, then X is compact.

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