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A GENERALIZATION OF AN INEQUALITY OF ZYGMUND

R. PERETZ

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

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ABSTRACT. The well known Bernstein inequality states that if D is a disk centered at the origin with radius R and if $p(z)$ is a polynomial of degree n, then $\max_{z\in D} |p'(z)| \le \frac{n}{R} \max_{z\in D} |p(z)|$ with equality iff $p(z) = AZ^n$. However it is true that we have the following better inequality:

$$\max_{z\in D} |p'(z)| \le \frac{n}{R} \max_{z\in D} |\mathrm{Re}\; p(z)|$$

with equality iff $p(z) = AZ^n$.

This is a consequence of a general equality that appears in Zygmund [7] (and which is due to Bernstein and Szegö): For any polynomial $p(z)$ of degree n and for any $1 \le p < \infty$ we have

$$\left\{ \int_0^{2\pi} |p'(e^{ix})|^p dx \right\}^{1/p} \le A_p n \left\{ \int_0^{2\pi} |\mathrm{Re}\; p(e^{ix})|^p dx \right\}^{1/p}$$

where $A_p^p = \pi^{1/2} \dfrac{\Gamma(\frac{1}{2} p + 1)}{\Gamma(\frac{1}{2} p + \frac{1}{2})}$ with equality iff $p(z) = AZ^n$.

In this note we generalize the last result to domains different from Euclidean disks by showing the following: If $g(e^{ix})$ is differentiable and if $p(z)$ is a polynomial of degree n then for any $1 \le p < \infty$ we have

$$\left\{\int_0^{2\pi} |g(e^{i\theta})p'(g(e^{i\theta}))|^p d\theta\right\}^{1/p} \le A_p n \max_{\beta}\left\{\int_0^{2\pi} |\mathrm{Re}\{p(e^{i\beta}g(e^{i\theta}))\}|^p d\theta\right\}^{1/p}$$

with equality iff $p(z) = Az^n$.

We then obtain some conclusions for Schlicht Functions.

Key Words and Phrases: Bernstein inequality, Bernstein-Szegö inequality, Krzyz problem, Dirichlet kernel, trigonometric interpolation

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1. INTRODUCTION.

The classical result of Bernstein as it appears in [2] is <u>Bernstein Inequality.</u> If D is a Euclidean disk and P is a polynomial of degree n over $\mathbb{C}$, then

$$\|p'\|_D \le \frac{n}{\mathrm{tr}(D)} \|p\|_D \tag{1}$$

where $\|f\|_D = \sup\limits_D |f(z)|$ and $\mathrm{tr}(D)$ is the transfinite diameter of D (which is the disk's radius in this case).

This result was generalized to various directions. The following theorem appears in [1]. Let $0 \le k \le 1$ and let E be a closed k-quasidisk, then

THEOREM. For any polynomial P of degree n we have

$$\left|\frac{p(z_1)-p(z_2)}{z_1-z_2}\right| \le c_1 \frac{n^{1+k}}{\mathrm{tr}(E)} \|p\|_E \,, \quad z_1, z_2 \in E \tag{2}$$

and

$$\|p'\|_E \le c_2 \frac{n^{1+k}}{\mathrm{tr}(E)} \|p\|_E \tag{3}$$

where $c_1 = 2^{-k}e(\frac{\pi}{4} + 1)$ and $c_2 = 2^{-k}e$.

Another direction of generalization arises naturally in the following:

Let β be the class of all analytic functions $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ in $|z| < 1$ such that $0 < |f(z)| < 1$. A problem posed by Krzyz [4] is to determine $A_n = \max\limits_{\beta} |a_n|$, $n \ge 1$ [3]. The conjecture (which is still unsolved) is that $A_n = \frac{2}{e}$ and that it is attained only by rotations of

$$g_n(z) = \exp\left(-\frac{z^n-1}{z^n+1}\right).$$

Let $f(z)$ be an extremal function for A_n.

CONJECTURE. $|f(0)| \le \frac{1}{e}$ and equality holds only for rotations of g_n.

A theorem which indicates that this conjecture may be true is:

THEOREM [5]. If $n = 2p + 1$ and if $a_1 = a_3 = \cdots = a_{2p-1} = 0$, then $|a_0| \le \frac{1}{e}$. Equality sign occurs iff $|a_n| = \frac{2}{e}$

The proof of this uses the following generalization of (1): Let $D(0,1) = \{z \in \mathbb{C} \mid |z| < 1\}$ and let p be any polynomial of degree n over $\mathbb{C}$, then

$$\|p'\|_{D(0,1)} \le n\|\mathrm{Re}\; p\|_{D(0,1)} \tag{4}$$

This follows from an inequality of Zygmund [7].

THEOREM. For any polynomial p of degree n and for any $1 \le p < \infty$ we have

$$\left\{ \int_0^{2\pi} |p'(e^{ix})|^p dx \right\}^{1/p} \le A_p n \left\{ \int_0^{2\pi} |\mathrm{Re}\, p(e^{ix})|^p dx \right\}^{1/p} \tag{5}$$

where

$$A_p^p = \pi^{1/2} \frac{\Gamma(\frac{1}{2} p + 1)}{\Gamma(\frac{1}{2} p + \frac{1}{2})} \tag{6}$$

and equality occurs in (5) iff $p(z) = Az^n$.

In this note we indicate a way to generalize (5) to domains E other than $D(0,1)$ by using the same ideas as in Zygmund's proof applied to $p \circ g$ where g is a quite general mapping $D(0,1) \to E$.

2. RESULTS.

THEOREM 1. Let g be a complex valued function of e^{ix}, $0 \le x \le 2\pi$. Suppose that $\{\arg g(e^{ix}) | 0 \le x \le 2\pi\} \supseteq [0, 2\pi/n]$ and that $\frac{dg(e^{ix})}{dx}$ exists, then for any non-negative, non- decreasing convex function χ , for any $\alpha \in \mathbb{R}$ and for any polynomial P of degree n over $\mathbb{C}$ we have

$$\int_0^{2\pi} \chi\left(n^{-1} \left| \mathrm{Im}\left\{e^{i\alpha} g(e^{i\theta}) p'(g(e^{i\theta}))\right\}\right|\right) d\theta \le \max_\beta \left\{\int_0^{2\pi} \chi\left(\left|\mathrm{Re}\{p(e^{i\beta} g(e^{i\theta}))\}\right|\right) d\theta\right\} \tag{7}$$

equality occurs in (7) iff $p(z) = Az$.

We remark that the consequences of Theorem 1 hold true even if the condition

$$\{\arg g(e^{ix}) | 0 \le x \le 2\pi\} \supseteq [0, 2\pi/n]$$

is dropped.

We will indicate at the end of Section 4 how to prove that.

With the notations of Theorem 1 we have

THEOREM 2. If $1 \le p < \infty$, then

$$\left\{\int_0^{2\pi} |g(e^{i\theta}) p'(g(e^{i\theta}))|^p d\theta\right\}^{1/p} \le A_p n \max_\beta \left\{\int_0^{2\pi} |\mathrm{Re}\{p(e^{i\beta} g(e^{i\theta}))\}|^p d\theta\right\}^{1/p} \tag{8}$$

with equality iff $p(z) = Az^n$.

As a consequence we derive an analogous theorem to (1),

THEOREM 3. If E is a simply connected domain such that $0 \in E$, and if $G : D(0,1) \to E$ is a Riemann mapping normalized by $G(0) = 0$, then for every $1 \le p < \infty$ and every $0 \le r < 1$ we have

$$\left\{\int_0^{2\pi} |P'(G(re^{i\theta}))|^p \, d\theta\right\}^{1/p} \le \frac{4 A_p n}{r|G'(0)|} \max_\beta \left\{\int_0^{2\pi} |\mathrm{Re}\{P(e^{i\beta} G(re^{i\theta}))\}|^p \, d\theta\right\}^{1/p} \tag{9}$$

This last inequality is not sharp.

Returning to the function g of Theorem 1 we add

COROLLARY.

$$\max_{\alpha}\left\{\int_0^{2\pi}\chi\left(\left|\operatorname{Im}\{e^{i\alpha}g(e^{i\theta})\}\right|\right)d\theta\right\} = \max_{\beta}\left\{\int_0^{2\pi}\chi\left(\left|\operatorname{Re}\{e^{i\beta}g(e^{i\theta})\}\right|\right)d\theta\right\} \tag{10}$$

$$\left\{\int_0^{2\pi}|g(e^{i\theta})|^p d\theta\right\}^{1/p} \le A_p \max_{\beta}\left\{\int_0^{2\pi}|\operatorname{Re}\{e^{i\beta}g(e^{i\theta})\}|^p d\theta\right\}^{1/p} \tag{11}$$

The last corollary can be seen directly, but, it shows that we cannot drop "max" on the right hand of the above inequalities since it is easy to find a g such that $\|\operatorname{Re} g\|_p \le 1$ while $\lim_{p\to\infty}\|g\|_p = \infty$.

3. PREPARATIONS.

Let $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial of degree n , where $c_0 \in \mathbb{R}$. We denote

$$S(z) = \frac{1}{2}(p(z) + \overline{p(z)}) \ , \ \tilde{S}(z) = \frac{1}{2i}(p(z) - \overline{p(z)}) \tag{12}$$

Let g be a complex valued function of e^{ix} , $x \in \mathbb{R}$ such that $\{\arg g(e^{ix}) | 0 \le x \le 2\pi) \supseteq [0,\frac{2\pi}{n}]$ and such that $\frac{dg}{dx}(e^{ix})$ exists. We denote

$$g(e^{ix}) = R(x)e^{i\phi(x)},\ R(x) = |g(e^{ix})| \ ,\ \phi(x) = \arg g(e^{ix}) \tag{13}$$

$$S(x,t) = C_0 + \sum_{\nu=1}^{n} R^{\nu}(x)(a_\nu \cos \nu t + b_\nu \sin \nu t) \tag{14}$$

$$\tilde{S}(x,t) = \sum_{\nu=1}^{n} R^{\nu}(x)(a_\nu \sin \nu t - b_\nu \cos \nu t)$$

$$\text{where } c_0, a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$$

where the coefficients a,b are such that

$$S(x,\phi(x)) = S(g(e^{ix})) \ ,\ \tilde{S}(x,\phi(x)) = \tilde{S}(g(e^{ix})) \ . \tag{15}$$

As in Zygmund we denote the modified Dirichlet kernel and it's conjugate kernel by $D_n^*(u)$, $\tilde{D}_n^*(u)$ respectively. Thus

$$D_n^*(u) = \frac{1}{2}\sum_{\nu=1}^{n-1}\cos \nu u + \frac{1}{2}\cos nu = \frac{\sin nu}{2\tan\frac{1}{2}u} \tag{16}$$

$$\tilde{D}_n^*(u) = \sum_{\nu=1}^{n-1}\sin \nu u + \frac{1}{2}\sin nu = (1 - \cos nu)\,\frac{1}{2}\cot\frac{1}{2}u \ .$$

We will also need the zeros of $\cos nt$

$$u_\nu = (2\nu-1)\pi/2n\ , \quad \nu = 1,2,\cdots,2n \tag{17}$$

$\phi_{2n}(t)$ will be a step function which has jumps $\frac{\pi}{n}$ at the points u_ν . By (3.6), (3.21) on pages 10, 11 [7] we have

THEOREM (Zygmund)

$$S(x,u) = a_n R^n(x)\cos nu + \frac{1}{\pi}\int_0^{2\pi} S(x,t)D_n^*(t-u)d\phi_{2n}(t) \tag{18}$$

$$\tilde{S}(x,u) = a_n R^n(x)\sin nu + \frac{1}{\pi}\int_0^{2\pi} S(x,t)\tilde{D}_n^*(t-u)d\phi_{2n}(t)$$

Thus for any real number α we have

$$S(g(e^{ix}))\cos\alpha - \tilde{S}(g(e^{ix}))\sin\alpha = a_n R^n(x)\cos[n\phi(x)+\alpha] + \tag{19}$$

$$+ \frac{1}{\pi}\int_0^{2\pi} S(x,t)\left\{\frac{\sin[n(\phi(x)-t)+\alpha]-\sin\alpha}{2\tan\frac{1}{2}(\phi(x)-t)}\right\} d\phi_{2n}(t)$$

4. A PROOF OF THEOREM 1.

As in Zygmund, let x_0 be a root of $\sin[n\phi(x)+\alpha]$ such that $\cos[n\phi(x_0)+\alpha] = 1$. We differentiate (19) with respect to x and substitute $x = x_0$. By (12) we have

$$\frac{dS}{dx}(g(e^{ix})) = -\mathrm{Im}\left\{e^{ix}g'(e^{ix})p'\left[g(e^{ix})\right]\right\} \tag{20}$$

$$\frac{d\tilde{S}}{dx}(g(e^{ix})) = \mathrm{Re}\left\{e^{ix}g'(e^{ix})p'\left[g(e^{ix})\right]\right\}$$

This takes care of the left hand side of (19) . On the right hand side we first differentiate $R(x)$ and use:

$$\frac{R'(x)}{R(x)} = -\mathrm{Im}\left\{\frac{e^{ix}g'(e^{ix})}{g(e^{ix})}\right\},$$

$$\frac{\partial}{\partial t}\{\tilde{S}(x,t)\} = \sum_{\nu=1}^{n} \nu R^\nu(x)(a_\nu\cos\nu t + b_\nu\sin\nu t)\ ,$$

$$\left.\frac{\partial\tilde{S}}{\partial t}\right|_{t=\phi(x)} = \mathrm{Re}\left\{g(e^{ix})p'\left[g(e^{ix})\right]\right\},$$

$$\left.\frac{\partial\tilde{S}}{\partial t}\right|_{t=\phi(x)} = \mathrm{Im}\left\{g(e^{ix})p'\left[g(e^{ix})\right]\right\},$$

$$-\mathrm{Im}\left\{\frac{e^{ix}g'(e^{ix})}{g(e^{ix})}\right\}\left\{\mathrm{Re}\left\{g(e^{ix})p'\left[g(e^{ix})\right]\right\}\cos\alpha - \mathrm{Im}\left\{g(e^{ix})p'\left[g(e^{ix})\right]\right\}\sin\alpha\right\} \tag{21}$$

We now differentiate $\phi(x)$ on the right hand side of (19). Using (3.22) on page 12 [7] we get

$$\mathrm{Re}\left\{\frac{e^{ix_0} g'(e^{ix_0})}{g(e^{ix_0})}\right\} \frac{1}{n} \sum_{\nu=1}^{2n} \frac{(-1)^{\nu+1} + \sin\alpha}{4\sin^2 \frac{1}{2}(\phi(x_0)-u_\nu)} S(x_0,u_\nu) \tag{22}$$

where we have used $\phi'(x_0) = \mathrm{Re}\left\{\frac{e^{ix_0} g'(e^{ix_0})}{g(e^{ix_0})}\right\}$.

Combining (20), (21), (22) with (19) gives

$$- \mathrm{Im}\left\{e^{i(x_0+\alpha)} g'(e^{ix_0})p'\left[g(e^{ix_0})\right]\right\} =$$

$$-\mathrm{Im}\left\{\frac{e^{ix_0} g'(e^{ix_0})}{g(e^{ix_0})}\right\} \mathrm{Re}\left\{e^{i\alpha}g(e^{ix_0})p'\left[g(e^{ix_0})\right]\right\} +$$

$$+ \mathrm{Re}\left\{\frac{e^{ix_0} g'(e^{ix_0})}{g(e^{ix_0})}\right\} \frac{1}{n} \sum_{\nu=1}^{2n} \frac{(-1)^{\nu+1} + \sin\alpha}{4\sin^2 \frac{1}{2}(\phi(x_0)-u_\nu)} S(x_0,u_\nu)$$

We now use the identity $\mathrm{Im}(A\cdot B) = \mathrm{Re}(A)\mathrm{Im}(B) + \mathrm{Im}(A)\mathrm{Re}(B)$ with

$A = \frac{e^{ix_0} g'(e^{ix_0})}{g(e^{ix_0})}$, $B = e^{i\alpha}g(e^{ix_0})p'\left[g(e^{ix_0})\right]$ and get finally

$$\mathrm{Im}\left\{e^{i\alpha}g(e^{ix_0})p'\left[g(e^{ix_0})\right]\right\} = - \frac{1}{n} \sum_{\nu=1}^{2n} \frac{(-1)^{\nu+1} + \sin\alpha}{4\sin^2 \frac{1}{2}(\phi(x_0)-u_\nu)} S(x_0,u_\nu) \tag{23}$$

This is a generalization of (3.22) on page 12 of [7]. Let

$$\beta_\nu = \left|\frac{(-1)^{\nu+1} + \sin\alpha}{4\sin^2 \frac{1}{2}(\phi(x_0)-u_\nu)}\right| , \quad \nu = 1,2,\cdots,2n \tag{24}$$

then

$$\beta_1 + \beta_2 + \cdots + \beta_{2n} = n^2 \tag{25}$$

We use (23) with $R(\theta + x - x_0)e^{i(\phi(\theta)+\phi(x)-\phi(x_0))}$ in place of $g(e^{ix})$ (see (13)) and get

$$\left|\mathrm{Im}\left\{e^{i\alpha}g(e^{i\theta})p'\left[g(e^{i\theta})\right]\right\}\right| \le \frac{1}{n} \sum_{\nu=1}^{2n} \beta_\nu \left|\mathrm{Re}\left\{P\left(e^{i(u_\nu-\phi(x_0))}g(e^{i\theta})\right)\right\}\right|$$

Using the assumptions on χ, (25) and applying Jensen's inequailty we get

$$\chi\left(n^{-1}\left|\mathrm{Im}\left\{e^{i\alpha}g(e^{i\theta})p'\left(g(e^{i\theta})\right)\right\}\right|\right) \le \frac{1}{n^2}\sum_{\nu=1}^{2n}\beta_\nu\chi\left(\left|\mathrm{Re}\left\{P\left(e^{i(u_\nu-\phi(x_0))}g(e^{i\theta})\right)\right\}\right|\right)$$

Integration with respect to θ gives (7) . The equality assertion follows from Zygmund. This completes the proof of Theorem 1. □

To prove that the consequence of Theorem 1 hold true even if we drop the condition

$$\{\arg g(e^{ix})|0 \le x \le 2\pi\} \supseteq [0,2\pi/n]$$

we can use (3,23) in [7] with the following

$$S(\theta) = c_0 + \sum_1^n \left(a_\nu \cos \nu\theta + b_\nu \sin \nu\theta\right)R^\nu \quad \text{where} \quad x_0 = -\frac{\alpha}{n}.$$

Then for $R \ge 0,\ 0 \le \theta,\ \alpha \le 2\pi$ we get

$$\left|\mathrm{Im}\left(e^{i\alpha}Re^{i\theta}p'\left(Re^{i\theta}\right)\right)\right| \le \frac{1}{n}\sum_1^{2n}\beta_\nu\left|\mathrm{Re}\ p\left(Re^{i\left(\theta+u_k+\frac{\alpha}{n}\right)}\right)\right|,$$

where the β_ν are independent of R,θ. From that we proceed as in the proof of Theorem 1.

5. A PROOF OF THEOREM 2.

Let $\chi(t) = t^p$ in (7) . We get

$$\int_0^{2\pi}\left|\mathrm{Im}\left\{e^{i\alpha}g(e^{i\theta})p'\left(g(e^{i\theta})\right)\right\}\right|^p d\theta \le n^p \max_\beta\left\{\int_0^{2\pi}\left|\mathrm{Re}\left\{P\left(e^{i\beta}g(e^{i\theta})\right)\right\}\right|^p d\theta\right\}$$

Let $g(e^{i\theta})p'\left(g(e^{i\theta})\right) = A(\theta) + iB(\theta)$ then we have

$$\int_0^{2\pi}|B(\theta)\cos\alpha + A(\theta)\sin\alpha|^p\, d\theta \le n^p\max_\beta\left\{\int_0^{2\pi}\left|\mathrm{Re}\left\{P\left(e^{i\beta}g(e^{i\theta})\right)\right\}\right|^p d\theta\right\}$$

As in Zygmund we integrate this with respect to α over $0 \le \alpha \le 2\pi$, change the order of integration on the left hand side and use

$$\int_0^{2\pi}|a\cos\alpha + b\sin\alpha|^p\, d\alpha = (a^2+b^2)^{p/2}\int_0^{2\pi}|\sin\alpha|^p\, d\alpha$$

to get

$$\left\{\int_0^{2\pi}\left|g(e^{i\theta})p'\left(g(e^{i\theta})\right)\right|^p d\theta\right\}^{1/p}$$

$$\le \left\{\frac{2\pi}{\int_0^{2\pi}|\sin\alpha|^p d\alpha}\right\}^{1/p} n \max_\beta\left\{\int_0^{2\pi}\left|\mathrm{Re}\left\{P\left(e^{i\beta}g(e^{i\theta})\right)\right\}\right|^p d\theta\right\}^{1/p}$$

this proves (8) and completes the proof of Theorem 2. □

6. PROOFS OF THEOREM 3 AND THE COROLLARY.

By the normalization $G(0) = 0$ we can use Theorem 2 with $g(e^{ix}) = G(re^{ix})$. We apply Koebe's $\frac{1}{4}$ -theorem [6] to get $\frac{r|G'(0)|}{4} \le |G(re^{i\theta})|$. This bounds the left hand side of (8) from below and proves (9). $\square$

(10) follows from (7) with $p(z) = z$ applied to g and to ig.

(11) follows from (8) with $p(z) = z$. $\square$

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