A GENERALIZATION OF AN INEQUALITY OF ZYGUND

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ABSTRACT. The well known Bernstein inequality states that if \( D \) is a disk centered at the origin with radius \( R \) and if \( p(z) \) is a polynomial of degree \( n \), then
\[
\max_{z \in D} |p'(z)| \leq \frac{n}{R} \max_{z \in D} |p(z)|
\]
with equality iff \( p(z) = A z^n \).

However it is true that we have the following better inequality:
\[
\max_{z \in D} |p'(z)| \leq \frac{n}{R} \max_{z \in D} \text{Re } p(z)
\]
with equality iff \( p(z) = A z^n \).

This is a consequence of a general equality that appears in Zygmund [7] (and which is due to Bernstein and Szegö): For any polynomial \( p(z) \) of degree \( n \) and for any \( 1 \leq p < \infty \) we have
\[
\left\{ \int_0^{2\pi} |p'(e^{ix})|^p dx \right\}^{1/p} \leq A_p n \left\{ \int_0^{2\pi} \text{Re } p(e^{ix})^p dx \right\}^{1/p}
\]
where
\[
A_p = \pi^{1/2} \frac{\Gamma\left(\frac{1}{2} p + 1\right)}{\Gamma\left(\frac{1}{2} p + \frac{1}{2}\right)}
\]
with equality iff \( p(z) = A z^n \).

In this note we generalize the last result to domains different from Euclidean disks by showing the following: If \( g(e^{ix}) \) is differentiable and if \( p(z) \) is a polynomial of degree \( n \) then for any \( 1 \leq p < \infty \) we have
\[
\left\{ \int_0^{2\pi} |g(e^{i\theta})p'(g(e^{i\theta}))|^p d\theta \right\}^{1/p} \leq A_p n \max_\beta \left\{ \int_0^{2\pi} \text{Re } (p(g(e^{i\theta}))^p d\theta \right\}^{1/p}
\]
with equality iff \( p(z) = A z^n \).

We then obtain some conclusions for Schlicht Functions.

Key Words and Phrases: Bernstein inequality, Bernstein-Szegö inequality, Krzyz problem, Dirichlet kernel, trigonometric interpolation

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1. INTRODUCTION.

The classical result of Bernstein as it appears in [2] is Bernstein Inequality. If \( D \) is a Euclidean disk and \( P \) is a polynomial of degree \( n \) over \( \mathbb{C} \), then

\[
\|p\|_D \leq \frac{n}{\text{tr}(D)} \|p\|_D
\]

where \( \|f\|_D = \sup_{D} |f(z)| \) and \( \text{tr}(D) \) is the transfinite diameter of \( D \) (which is the disk's radius in this case).

This result was generalized to various directions. The following theorem appears in [1]. Let \( 0 \leq k \leq 1 \) and let \( E \) be a closed \( k \)-quasidisk, then

**THEOREM.** For any polynomial \( P \) of degree \( n \) we have

\[
\left| \frac{p(z_1) - p(z_2)}{z_1 - z_2} \right| \leq c_1 \frac{n^{1+k}}{\text{tr}(E)} \|p\|_E, \quad z_1, z_2 \in E
\]

and

\[
\|p\|_E \leq c_2 \frac{n^{1+k}}{\text{tr}(E)} \|p\|_E
\]

where \( c_1 = 2^{-k} e \left( \frac{\pi}{4} + 1 \right) \) and \( c_2 = 2^{-k} e \).

Another direction of generalization arises naturally in the following:

Let \( \beta \) be the class of all analytic functions \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) in \( |z| < 1 \) such that \( 0 < |f(z)| < 1 \). A problem posed by Krzyz [4] is to determine \( A_n = \max_{\beta} |a_n|, \ n \geq 1 \) [3]. The conjecture (which is still unsolved) is that

\[
A_n = \frac{2}{e}
\]

and that it is attained only by rotations of

\[ g_n(z) = \exp \left( -\frac{z^{n-1}}{z^n+1} \right). \]

Let \( f(z) \) be an extremal function for \( A_n \).

**CONJECTURE.** \( |f(0)| \leq \frac{1}{e} \) and equality holds only for rotations of \( g_n \).

A theorem which indicates that this conjecture may be true is:

**THEOREM [5].** If \( n = 2p + 1 \) and if \( a_1 = a_3 = \cdots = a_{2p-1} = 0 \), then

\[
|a_0| \leq \frac{1}{e}. \quad \text{Equality sign occurs iff } |a_n| = \frac{2}{e}
\]

The proof of this uses the following generalization of (1): Let \( D(0,1) = \{ z \in \mathbb{C} \mid |z| < 1 \} \) and let \( p \) be any polynomial of degree \( n \) over \( \mathbb{C} \), then

\[
\|p\|_{D(0,1)} \leq n \|\text{Re } p\|_{D(0,1)}
\]

This follows from an inequality of Zygmund [7].

**THEOREM.** For any polynomial \( p \) of degree \( n \) and for any \( 1 \leq p < \infty \) we have
where
\[ A_p^n = \pi^{1/2} \frac{\Gamma\left(\frac{1}{p} + 1\right)}{\Gamma\left(\frac{1}{p} + \frac{1}{2}\right)} \]

and equality occurs in (5) iff \( p(z) = Az^n \).

In this note we indicate a way to generalize (5) to domains \( E \) other than \( D(0,1) \) by using the same ideas as in Zygmund's proof applied to \( p \circ g \) where \( g \) is a quite general mapping \( D(0,1) \to E \).

2. RESULTS.

THEOREM 1. Let \( g \) be a complex valued function of \( e^{ix}, 0 \leq x \leq 2\pi \). Suppose that \( \{\arg g(e^{ix})|0 \leq x \leq 2\pi\} \supseteq [0,2\pi/n] \) and that \( \frac{dg(e^{ix})}{dx} \) exists, then for any non-negative, non-decreasing convex function \( \chi \), for any \( \alpha \in \mathbb{R} \) and for any polynomial \( P \) of degree \( n \) over \( \mathbb{C} \) we have
\[
\left( \int_0^{2\pi} \chi^{-1}\left( |\text{Im}(e^{i\alpha}g(e^{i\theta})p'(g(e^{i\theta})))| \right) d\theta \right)^{1/p} \leq \max_\beta \left( \int_0^{2\pi} \chi^{-1}\left( |\text{Re}(p(e^{i\beta}g(e^{i\theta})))| \right) d\theta \right)^{1/p}
\]

equality occurs in (7) iff \( p(z) = Az^n \).

We remark that the consequences of Theorem 1 hold true even if the condition
\( \{\arg g(e^{ix})|0 \leq x \leq 2\pi\} \supseteq [0,2\pi/n] \)
is dropped.

We will indicate at the end of Section 4 how to prove that.

With the notations of Theorem 1 we have

THEOREM 2. If \( 1 \leq p < \infty \), then
\[
\left( \int_0^{2\pi} |g(e^{i\theta})p'(g(e^{i\theta})))|^p d\theta \right)^{1/p} \leq A_p^n \max_\beta \left( \int_0^{2\pi} |\text{Re}(p(e^{i\beta}g(e^{i\theta})))|^p d\theta \right)^{1/p}
\]

with equality iff \( p(z) = Az^n \).

As a consequence we derive an analogous theorem to (1),

THEOREM 3. If \( E \) is a simply connected domain such that \( 0 \in E \), and if \( G : D(0,1) \to E \) is a Riemann mapping normalized by \( G(0) = 0 \), then for every \( 1 \leq p < \infty \) and every \( 0 \leq r < 1 \) we have
\[
\left( \int_0^{2\pi} |P'(G(re^{i\theta}))|^p d\theta \right)^{1/p} \leq \frac{4 A_p^n}{r|G'(0)|} \max_\beta \left( \int_0^{2\pi} |\text{Re}(P(e^{i\beta}G(re^{i\theta})))|^p d\theta \right)^{1/p}
\]

This last inequality is not sharp.
Returning to the function $g$ of Theorem 1 we add

**COROLLARY.**

$$
\max_{\alpha} \left\{ \frac{2\pi}{\alpha} \int_0^{2\pi} \left| \frac{\ln(\frac{1}{\alpha} g(e^{i\theta}))}{\ln g(e^{i\theta})} \right| d\theta \right\} = \max_{\beta} \left\{ \frac{2\pi}{\beta} \int_0^{2\pi} \left| \frac{\ln(\frac{1}{\beta} g(e^{i\theta}))}{\ln g(e^{i\theta})} \right| d\theta \right\} \tag{10}
$$

$$
\left\{ \int_0^{2\pi} \frac{|g(e^{i\theta})|^n}{|P|} d\theta \right\}^{1/p} \leq A_p \max_{\beta} \left\{ \frac{2\pi}{\beta} \int_0^{2\pi} \left| \frac{\ln(\frac{1}{\beta} g(e^{i\theta}))}{\ln g(e^{i\theta})} \right| d\theta \right\}^{1/p} \tag{11}
$$

The last corollary can be seen directly, but, it shows that we cannot drop "max" on the right hand of the above inequalities since it is easy to find a $g$ such that $\| \text{Re} g \|_p \leq 1$ while $\lim_{p \to \infty} \| g \|_p = \infty$.

**3. PREPARATIONS.**

Let $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial of degree $n$, where $c_0 \in \mathbb{R}$. We denote

$$
S(z) = \frac{1}{2}(p(z) + \overline{p(z)}) \quad \tilde{S}(z) = \frac{1}{2i}(p(z) - \overline{p(z)}) \tag{12}
$$

Let $g$ be a complex valued function of $e^{ix}$, $x \in \mathbb{R}$ such that

$\{ \arg g(e^{ix}) \} |0 \leq x \leq 2\pi| \geq 0, \frac{2\pi}{n}$

and such that $\frac{dg(e^{ix})}{dx}$ exists. We denote

$$
g(e^{ix}) = R(x)e^{i\phi(x)}, \quad R(x) = |g(e^{ix})|, \quad \phi(x) = \arg g(e^{ix}) \tag{13}
$$

$$
S(x,t) = c_0 + \sum_{\nu=1}^{n} R(x)(a_\nu \cos \nu t + b_\nu \sin \nu t) \tag{14}
$$

$$
\tilde{S}(x,t) = \sum_{\nu=1}^{n} R(x)(a_\nu \sin \nu t - b_\nu \cos \nu t)
$$

where $c_0, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$.

where the coefficients $a, b$ are such that

$$
S(x, \phi(x)) = S(g(e^{ix})), \quad \tilde{S}(x, \phi(x)) = \tilde{S}(g(e^{ix})) \tag{15}
$$

As in Zygmund we denote the modified Dirichlet kernel and it's conjugate kernel by $D_n^*(u), \tilde{D}_n^*(u)$ respectively. Thus

$$
D_n^*(u) = \frac{1}{2} \sum_{\nu=1}^{n-1} \cos \nu u + \frac{1}{2} \cos nu = \frac{\sin nu}{2 \tan \frac{1}{2} u} \tag{16}
$$

$$
\tilde{D}_n^*(u) = \sum_{\nu=1}^{n-1} \sin \nu u + \frac{1}{2} \sin nu = (1 - \cos nu) \frac{1}{2} \cot \frac{1}{2} u.
$$

We will also need the zeros of $\cos nt$
u_\nu = (2\nu-1)\pi/2n, \nu = 1, 2, \ldots, 2n \tag{17}

2n \tau_k \tau_v \tau_1,2,\ldots,2n

\phi_{2n}(t) will be a step function which has jumps \frac{\pi}{n} at the points \tau_\nu. By (3.6), (3.21) on pages 10, 11 \cite{7} we have

**THEOREM (Zygmund)**

\[ S(x, u) = a_n R^n(x) \cos \nu u + \frac{1}{\pi} \int_0^{2\pi} S(x, t) D_n(t-u) d\phi_{2n}(t) \tag{18} \]

\[ \tilde{S}(x, u) = a_n R^n(x) \sin \nu u + \frac{1}{\pi} \int_0^{2\pi} S(x, t) D_n(t-u) d\phi_{2n}(t) \]

Thus for any real number \alpha we have

\[ S(g(e^{ix})) \cos \alpha - \tilde{S}(g(e^{ix})) \sin \alpha = a_n R^n(x) \cos [n \phi(x) + \alpha] + \]

\[ \frac{1}{\pi} \int_0^{2\pi} S(x, t) \left\{ \frac{\sin [n \phi(x) - t] + \alpha - \sin \alpha}{2 \tan \frac{n}{2} (\phi(x) - t)} \right\} d\phi_{2n}(t) \tag{19} \]


As in Zygmund, let \phi_0 be a root of \sin [n \phi(x) + \alpha] such that \cos [n \phi(x_0) + \alpha] = 1. We differentiate (19) with respect to \nu and substitute \nu = \phi_0. By (12) we have

\[ \frac{dS}{dx}(g(e^{ix})) = -\text{Im} \left\{ e^{ix} g'(e^{ix}) p \left( g(e^{ix}) \right) \right\} \tag{20} \]

\[ \frac{dS}{dx}(g(e^{ix})) = \text{Re} \left\{ e^{ix} g'(e^{ix}) p \left( g(e^{ix}) \right) \right\} \]

This takes care of the left hand side of (19). On the right hand side we first differentiate \( R(x) \) and use:

\[ R'(x) = -\text{Im} \left\{ \frac{e^{ix} g'(e^{ix}) p}{g(e^{ix})} \right\}, \]

\[ \frac{d}{dt} \tilde{S}(x, t) = \sum_{\nu=1}^{n} \nu R^{\nu}(x) (a_\nu \cos \nu t + b_\nu \sin \nu t), \]

\[ \left. \frac{d\tilde{S}}{dt} \right|_{t=\phi(x)} = \text{Re} \left\{ g(e^{ix}) p' \left( g(e^{ix}) \right) \right\}, \]

\[ \left. \frac{d\tilde{S}}{dt} \right|_{t=\phi(x)} = \text{Im} \left\{ g(e^{ix}) p' \left( g(e^{ix}) \right) \right\}, \]

\[ -\text{Im} \left\{ \frac{e^{ix} g'(e^{ix})}{g(e^{ix})} \right\} \left\{ \text{Re} \left( g(e^{ix}) p' \left( g(e^{ix}) \right) \right) \cos \alpha - \text{Im} \left( g(e^{ix}) p' \left( g(e^{ix}) \right) \right) \sin \alpha \right\} \tag{21} \]
We now differentiate $\phi(x)$ on the right hand side of (19). Using (3.22) on page 12 [7] we get

$$\text{Re}\left\{ \frac{\text{i}x_0}{g(e^{\text{i}x_0})} \cdot \frac{\text{i}x_0}{g(e^{\text{i}x_0})} \right\} \cdot \frac{1}{\text{i}n} \sum_{\nu=1}^{2n} \frac{\nu+1 - \sin \alpha}{\sin \frac{\nu}{2} (\phi(x_0) - u_\nu)} S(x_0, u_\nu)$$  \hfill (22)

where we have used $\phi'(x_0) = \text{Re}\left\{ \frac{\text{i}x_0}{g(e^{\text{i}x_0})} \right\}$.

Combining (20), (21), (22) with (19) gives

$$-\text{Im}\left\{ \frac{\text{i}x_0}{g(e^{\text{i}x_0})} \cdot \frac{\text{i}x_0}{g(e^{\text{i}x_0})} \right\} \cdot \frac{1}{\text{i}n} \sum_{\nu=1}^{2n} \frac{\nu+1 - \sin \alpha}{\sin \frac{\nu}{2} (\phi(x_0) - u_\nu)} S(x_0, u_\nu)$$  \hfill (23)

We now use the identity $\text{Im}(A \cdot B) = \text{Re}(A)\text{Im}(B) + \text{Im}(A)\text{Re}(B)$ with

$$A = \frac{\text{i}x_0}{g(e^{\text{i}x_0})}, \quad B = e^{\text{i}\alpha} g(e^{\text{i}x_0})p'(g(e^{\text{i}x_0}))$$

and get finally

$$\text{Im}\left\{ e^{\text{i}\alpha} g(e^{\text{i}x_0})p'(g(e^{\text{i}x_0})) \right\} = -\frac{1}{\text{i}n} \sum_{\nu=1}^{2n} \frac{\nu+1 - \sin \alpha}{\sin \frac{\nu}{2} (\phi(x_0) - u_\nu)} S(x_0, u_\nu)$$  \hfill (24)

This is a generalization of (3.22) on page 12 of [7]. Let

$$\beta_\nu = \left| \frac{(-1)^{\nu+1} + \sin \alpha}{4 \sin^2 \frac{\nu}{2} (\phi(x_0) - u_\nu)} \right|, \quad \nu = 1, 2, \cdots, 2n$$  \hfill (25)

then

$$\beta_1 + \beta_2 + \cdots + \beta_{2n} = n^2$$

We use (23) with $R(\theta + x - x_0)e^{\text{i}(\phi(\theta)+\phi(x) - \phi(x_0))}$ in place of $g(e^{\text{i}x})$ (see (13)) and get

$$|\text{Im}\left\{ e^{\text{i}\alpha} g(e^{\text{i}\theta})p'(g(e^{\text{i}\theta})) \right\}| = \frac{1}{\text{i}n} \sum_{\nu=1}^{2n} \beta_\nu |\text{Re}\left\{ e^{\text{i}(u_\nu - \phi(x_0))} g(e^{\text{i}\theta}) \right\}|$$

Using the assumptions on $\chi$, (25) and applying Jensen's inequality we get
\[
\chi^{-1}\left|\text{Im}\left(e^{i\alpha}g(e^{i\theta})p\left(g(e^{i\theta})\right)\right)\right| \leq \frac{1}{n^2} \sum_{\nu=1}^{2n} \beta_\nu \left|\text{Re}\left(P\left(e^{i\phi(x_0)}g(e^{i\theta})\right)\right)\right|
\]

Integration with respect to \( \theta \) gives (7). The equality assertion follows from Zygmund. This completes the proof of Theorem 1. \( \Box \)

To prove that the consequence of Theorem 1 hold true even if we drop the condition

\[
\{\text{arg } g(e^{ix})|0 \leq x \leq 2\pi\} \subseteq [0,2\pi/n]
\]

we can use (3,23) in (7) with the following

\[
S(\theta) = c_0 + \sum_{\nu} \left(a_\nu \cos \nu \theta + b_\nu \sin \nu \theta\right)R^\nu \quad \text{where} \quad x_0 = -\frac{\alpha}{n}.
\]

Then for \( R \geq 0, \quad 0 \leq \theta, \quad \alpha \leq 2\pi \) we get

\[
\left|\text{Im}\left(e^{i\alpha}R^{-\theta}p\left(R^{-i\theta}\right)\right)\right| \leq \frac{1}{n} \sum_{\nu=1}^{2n} \beta_\nu \left|\text{Re}\left(P\left(e^{i\phi(x_0)}R^{-i\theta}\right)\right)\right|
\]

where the \( \beta_\nu \) are independent of \( R, \theta \). From that we proceed as in the proof of Theorem 1.

5. A PROOF OF THEOREM 2.

Let \( x(t) = e^P \) in (7). We get

\[
\int_0^{2\pi} \left|\text{Im}\left(e^{i\alpha}g(e^{i\theta})p\left(g(e^{i\theta})\right)\right)\right|^p d\theta = n^p \max_{\beta} \left\{\int_0^{2\pi} \left|\text{Re}\left(P\left(e^{i\beta}g(e^{i\theta})\right)\right)\right|^p d\theta\right\}
\]

Let \( g(e^{i\theta})p\left(g(e^{i\theta})\right) = A(\theta) + iB(\theta) \) then we have

\[
\int_0^{2\pi} |B(\theta)\cos \alpha + A(\theta)\sin \alpha|^p d\theta \leq n^p \max_{\beta} \left\{\int_0^{2\pi} \left|\text{Re}\left(P\left(e^{i\beta}g(e^{i\theta})\right)\right)\right|^p d\theta\right\}
\]

As in Zygmund we integrate this with respect to \( \alpha \) over \( 0 \leq \alpha \leq 2\pi \), change the order of integration on the left hand side and use

\[
\int_0^{2\pi} |a \cos \alpha + b \sin \alpha|^p d\alpha = (a^2+b^2)^{p/2} \int_0^{2\pi} |\sin \alpha|^p d\alpha
\]

to get

\[
\left\{\int_0^{2\pi} \left|g(e^{i\theta})p\left(g(e^{i\theta})\right)\right|^p d\theta\right\}^{1/p} \leq \left\{\int_0^{2\pi} \left|\text{Im}\left(e^{i\alpha}g(e^{i\theta})\right)\right|^p d\theta\right\}^{1/p}
\]

\[
\leq \left\{\int_0^{2\pi} \left|\sin \alpha\right|^p d\alpha\right\}^{1/p} \left\{\int_0^{2\pi} \left|\text{Re}\left(P\left(e^{i\beta}g(e^{i\theta})\right)\right)\right|^p d\theta\right\}^{1/p}
\]

this proves (8) and completes the proof of Theorem 2. \( \Box \)
6. PROOFS OF THEOREM 3 AND THE COROLLARY.

By the normalization \( G(0) = 0 \) we can use Theorem 2 with
\[ g(e^{i\lambda}) = G(re^{i\lambda}). \]
We apply Koebe's \( \frac{1}{4} \)-theorem [6] to get
\[ \frac{1}{4} |G'(0)| \leq |G(\text{re}^{i\theta})|. \]
This bounds the left hand side of (8) from below and proves (9).

(10) follows from (7) with \( p(z) = z \) applied to \( g \) and to \( \text{ig} \).

(11) follows from (8) with \( p(z) = z \).

REFERENCES


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