Internat. J. Math. & Math. Sci.
VOL. 16 NO. 2 (1993) 301-310

IDEMPOTENT AND COMPACT MATRICES ON LINEAR LATTICES: A SURVEY OF SOME LATTICE RESULTS AND RELATED SOLUTIONS OF FINITE RELATIONAL EQUATIONS

FORTUNATA LIGUORI, GIULIA MARTINI
and SALVATORE SESSA

Università di Napoli, Facoltà di Architettura, Istituto Matematico
Via Monteoliveto 3, 80134 Napoli, Italy

(Received February 4, 1991)

ABSTRACT. After a survey of some known lattice results, we determine the greatest idempotent (resp. compact) solution, when it exists, of a finite square rational equation assigned over a linear lattice. Similar considerations are presented for composite relational equations.

KEY WORDS AND PHRASES: Compact matrix, Idempotent matrix, Transitive matrix, Square finite relation equation, Residuated lattice.
1991 AMS SUBJECT CLASSIFICATION CODES. 15A24.

1. INTRODUCTION.

The theory of fuzzy relation equations is a powerful tool for applicational purposes like Fuzzy Control and Systems [13], Knowledge Engineering [5], Extremal Linear Programming [22].

We recall some well known definitions [5]. Throughout this paper, $L=(L,\wedge,\vee,\leq,0,1)$ is a (not necessarily complete) linear lattice with universal bounds $0,1,X=\{x_1,x_2,\cdots,x_n\}$ be a finite referential set, $A,B:X\to L$ be vectors (fuzzy sets) and $R:X\times X\to L$ be a (fuzzy) matrix such that

$$R\ o\ A=B, \tag{1.1}$$

where "o" is the max-min composition. In terms of membership functions, Eq. (1.1) is read as

$$\bigvee_{i=1}^{n}(A_i\ \wedge\ R_{ij})=B_j$$

for any $J\in I_n=\{1,2,\cdots,n\}$, where, for brevity of notation, we put $A(x_i)=A_i, B(x_j)=B_j, R(x_i,x_j)=R_{ij}$ for any $i,j\in I_n$.

A natural extension of the Eq. (1.1) is the following:

$$R\ o\ Q=T, \tag{1.2}$$

i.e.,

$$\bigvee_{j=1}^{n}(Q_{ij}\ \wedge\ R_{jk})=T_{ik}$$

for any $i,k\in I_n$, where $R,Q,T:X\times X\to L$ are matrices. As usual, if R is a matrix, we put $R^1=R, R^{n+1}=R^n oR$ for $n\geq 1$.

Following the symbology of Sanchez [5, 15], let α be the operation of residuation in L, i.e., $a\alpha b$ is the relative pseudocomplement of a in b defined by $a\alpha b=\sup\{c\in L: a\wedge c\leq b\}=1$ if $a\leq b$, $a\alpha b=b$ if $a>b$ for any $a,\ b\in L$. If A and B (resp. Q and T) are assigned in Eq. (1.1) (resp. (1.2)), let $\Re=\Re(A,B)$ (resp. $S=S(Q,T)$) be the set of all the square matrices R which solve the given equation. Sanchez [5] (cfr. also Luce [12] and Rudeanu [14] for Boolean equations, Zimmermann [22] and Di Nola and Lettieri [4] if L is, more generally, a complete right-residuated lattice and if X is infinite) determines, if $\Re$ (resp. S) is nonempty, the greatest (with respect to the partial ordering pointwise induced naturally in $\Re$ (resp. S) from the total ordering of L) element of $\Re$ (resp. S) defining the matrix $S = A\alpha B$ (resp. $S^* = Q^{-1}\alpha T$, where Q^{-1} is the transpose of Q): $X \times X \to L$ as

$$S_{ij} = (A\alpha B)_{ij} = A_i \alpha B_j$$

resp.

$$S^*_{jk} = (Q^{-1}\alpha T)_{jk} = \bigwedge_{i=1}^{n} (Q^{-1}_{ji}\alpha T_{ik}) = \bigwedge_{i=1}^{n} (Q_{ij}\alpha T_{ik}),$$

for any $i,j,k \in I_n$. The concept of max-min transitivity is the most widely used one for investigating properties of finite square Boolean matrices [10]. For square matrices over L, we say that R is max-min transitive if $R^2 \leq R$ (i.e., $R_{ih} \wedge R_{hj} \leq R_{ij}$ for any $i,h,j \in I_n$) and this notion is also widely dealt in several areas of research like clustering technology [5], information retrieval [20], preference relations [7, 8]. Moreover, if L is linear, Eq. (1.1) can be seen as a description of a finite-state system in which R represents a transition relationship between the assigned input A and the given output B. Following Kolodziejczyk [11] (cfr. also [3]), applying k times the transition, let $B^{(k)}$ be the resulting output defined by $B^{(k)} = R^k \; o \; A$ (assume, of course, $B^{(1)} = B$). The problem is to determine a solution $R \in \Re$, with an assigned type of transitivity, which guarantees the necessary speed of convergence (R is convergent if $R^{k+1} = R^k$ for some integer k). Similar considerations can be made on Eq. (1.2).

The authors of [3] and [11] have proved that in $\Re$ (resp. S) exist elements with different type of transitivity, mainly in $\Re$ (resp. S) were entirely characterized the max-min transitive elements (in particular, those having Schein rank equal to 1), determining the greatest and the minimal ones. Here we prove the existence in $\Re$, when nonempty, of compact (in particular, idempotent) elements of $\Re$ (R is compact if $R^2 \geq R$, R is idempotent if $R^2 = R$).

For each of them, a specific rule of convergence of their powers holds, so giving further information on the speed of convergence of the entire system. Related but different considerations are presented for the analogous elements of S.

2. A SURVEY OF SOME LATTICE RESULTS.

We refer to Birkhoff [1] for terminology of lattice theory. Now we recall some well known facts.

If $P,Q,R: X \times X \to L$ are assigned matrices, $P \leq R$ means $P_{ij} \leq R_{ij}$ for any $i,j \in I_n$ in L, $P < R$ means $P \leq R$ and $P \neq R$, $(P \wedge R)$, $(P \vee R): X \times X \to L$ are matrices pointwise defined as $(P \wedge R)_{ij} = P_{ij} \wedge R_{ij}$, $(P \vee R)_{ij} = P_{ij} \vee R_{ij}$ for any $i,j \in I_n$, similarly it is defined the infimum and the supremum of any finite set of matrices. It is well known that the max-min composition (1.2) is associative and the following properties hold:

$$(P \vee Q) \; o \; R = (P \; o \; Q) \vee (P \; o \; R) \text{ and } R \; o \; (P \vee Q) = (R \; o \; P) \vee (R \; o \; Q) \tag{2.1}$$

since L is linear and hence distributive.

Let U be the $n \times n$ identity matrix, i.e., $U_{ij} = 0$ if $i \neq j, U_{ii} = 1$ for any $i,j \in I_n$ and ϑ (resp. 1) be the $n \times n$ null (resp. unit) matrix. It is known that the structure $\mathfrak{F} = (\mathfrak{F}, \wedge, \vee, \leq, o, U, \vartheta, 1)$ of all the matrices $R: X \times X \to L$ is, in virtue of (2.1) and Thm. 3 and Thm. 4 of Sanchez [15], a bounded residuated l-monoid [1, p. 325], i.e., $(\mathfrak{F}, \wedge, \vee, \leq, U, \vartheta, 1)$ is a bounded residuated lattice and $(\mathfrak{F}, \leq, o, U)$ is a po-monoid [1, p. 319] with unit U, being complete if L is complete. In accordance to Sanchez [15], we can write that $Q^{-1}\alpha \; T = \sup\{R \in \mathfrak{F} : R \; o \; Q \leq T\}$ and $(R \; \alpha \; T^{-1})^{-1} = \sup \{Q \in \mathfrak{F} : R \text{ o } Q \leq T\}$.

REMARK 2.1. Since X is finite, the results of Sanchez [15] hold in the more weak hypothesis that L is a Brouwerian lattice, but it is easy to see that, under this last hypothesis, $\mathfrak{F}$ continues to be a bounded residuated l-monoid.

Following Shmuely [18], let $\mathfrak{C} = \{R \in \mathfrak{F}: R \leq R^2\}$ be the set of all non-negative (or compact) elements of $\mathfrak{F}$ and, as denoted usually, $\mathfrak{R}_\wedge$ be the set of all non-positive (or max-min transitive or subidempotent [1, p. 328]) elements of $\mathfrak{F}$. For any $R_1, R_2 \in \mathfrak{C}$, we have $(R_1 \vee R_2)^2 \geq R_1^2 \vee R_2^2 \geq R_1 \vee R_2$, so that $\mathfrak{C}$ is a join-subsemilattice (under $\vee$) of $\mathfrak{F}$. Dually, we have that $\mathfrak{R}_\wedge$ is meet-subsemilattice (under $\wedge$) of $\mathfrak{F}$.

REMARK 2.2. If L is complete, then $\mathfrak{F}$ is complete and consequently $\mathfrak{C}$ (resp. $\mathfrak{R}_\wedge$) is a complete join- (resp. meet-) subsemilattice of $\mathfrak{F}$ by Lemma 1 of Shmuely [18].

However, as it is known, $\mathfrak{R}_\wedge$ becomes a lattice defining the sup operation "$\bigcup$" as $R_1 \bigcup R_2 = \overline{R_1 \vee R_2}$, where $R_1, R_2 \in \mathfrak{R}_\wedge$ and "$\_$" stands for the max-min transitive closure of any matrix $R \in \mathfrak{F}$ defined as $\overline{R} = R \vee R^2 \vee \cdots \vee R^n$.

REMARK 2.3. If L is complete, then $\mathfrak{F}$, being a complete residuated lattice, satisfies properties (J_1) and (J_2) of Shmuely [18] (cfr. also Prop. 2.1 of [5]). Thus, by Corollary of [18], the smallest element $\overline{R} \in \mathfrak{R}_\wedge$ including $R \in \mathfrak{F}$ is given by $\overline{R} = R \vee R^2 \vee \cdots \vee R^n \vee \cdots$. However, it is proved, like in Kaufmann [9, p.95], that $\overline{R} = R \vee R^2 \vee \cdots \vee R^n$. This result holds for Boolean matrices ([10, Prop. 5.4.1], [16, Lemma 1.2]) too.

The study of the powers of a square Boolean matrix is useful in automata theory, information theory, etc., (e.g., [10]). The sequence $R, R^2, R^3, \cdots$ depends on two parameters p (the period of R) and k (the index of convergence or R), p and k being the smallest positive integers such that $R^{k+p} = R^k$. For Boolean matrices, these indices were widely studied (e.g., see Shao and Li [17] and references therein). These parameters can be defined also for a matrix $R \in \mathfrak{F}$ which either converges (i.e., $p = 1$) to an idempotent matrix or oscillates with finite period [21]. If $C \in \mathfrak{C}$, then $C^n = C^{n-1}$ [21] and $R^{n+1} = R^n$ if $R \in \mathfrak{R}_\wedge$ [6], thus we use these simple facts to prove the following:

THEOREM 2.4. The set $\xi = \mathfrak{C} \cap \mathfrak{R}_\wedge = \{R \in \mathfrak{F}: R = R^2\}$ of all the idempotent elements of $\mathfrak{F}$ is a lattice under the partial ordering induced by $\mathfrak{F}$ in ξ and the operations:

$$I \mathbin{\dot\wedge} J = (I \wedge J)^n \text{ and } (I \mathbin{\dot\vee} J)^{n-1} = \overline{I \vee J}, \tag{2.2}$$

where $I, J \in \xi$.

PROOF. For any $I, J \in \xi$, from above, we have that $(I \wedge J)^n$ and $(I \vee J)^{n-1}$ are elements of ξ. Prove that $(I \wedge J)^n = g.l.b\{I, J\}$ and, indeed, if $E \in \xi$ is such that $E \leq I$ and $E \leq J$, then $E \leq I \wedge J$ and thus $E = E^n \leq (I \wedge J)^n$. Dually, it is seen that $(I \wedge J)^{n-1} = l.u.b.\{I, J\}$.

REMARK 2.5. If L is complete, then it is easily seen that $\xi = (\xi, \wedge, \vee, \leq, \vartheta, 1)$ is a bounded complete lattice under the same operations (2.2) (cfr. [18, Thm. 4]).

Theorem 2.4 is an extension of a well known result (cfr. [19], Thm. 5.4.3 of Kim [10, p. 241)] concerning Boolean matrices.

REMARK 2.6. We note, in virtue of the first equality in (2.1), that $\mathfrak{R}$ (resp. S) is a join-subsemilattice (under $\vee$) of $\mathfrak{F}$, but generally it is not a meet-subsemilattice (under $\wedge$) of $\mathfrak{F}$ (cfr. Example 2.3 of [5]). Further, $\mathfrak{R}$ (resp. S) is a convex subset of $\mathfrak{F}$, i.e., if $R_1, R_2 \in \mathfrak{R}$ (resp. S) then $[R_1, R_2] = \{R \in \mathfrak{F}: R_1 \leq R \leq R_2\} \subseteq \mathfrak{R}$ (resp. S).

3. ON IDEMPOTENT SOLUTIONS OF EQ. (1.1).

Concerning Eq. (1.1), it was proved in [3] that the set $\mathfrak{T} = \mathfrak{T}(A, B) = \mathfrak{R} \cap \mathfrak{R}_\wedge$ of all max-min transitive solutions is nonempty iff $\mathfrak{R} \neq \emptyset$, $W \in \mathfrak{T}$ being the matrix, defined as $W_{ij} = B_j$ if $B_i > B_j$ and $W_{ij} = S_{ij}$ if $B_i \leq B_j$ for any $i, j \in I_n$, the greatest element of $\mathfrak{T}$, i.e., $W \geq R$ for any $R \in \mathfrak{T}$.

Let $\mathfrak{I} = \mathfrak{R} \cap \xi$ be the set of all idempotent solutions of Eq. (1.1). Of course, $\mathfrak{I} \subseteq \mathfrak{F}$ and let $B \in \mathfrak{F}$ be defined as $B_{ij} = B_j$ for any $i, j \in I_n$. Then $B \in \mathfrak{I}$ and hence $B = B^n \leq W^n \leq W$, i.e., the matrix $I = W^n$ belongs to $\mathfrak{R}$ by Remark 2.6. Since $I = W^n = W^{n+1} \in \xi$ [6], we deduce that $I \in \mathfrak{I}$ and further, $R = R^n \leq W^n = I$ for any $R \in \mathfrak{I}$. Thus, we have proved that

THEOREM 3.1. $\mathfrak{R} \neq \Phi$ iff $I \in \mathfrak{I}$. Further, $I \geq R$ for any $R \in \mathfrak{I}$.

REMARK 3.2. It is easily seen that an alternative definition of W is $W = S \wedge (B\alpha B)$, where $(B\alpha B) \in \mathfrak{F}$ is defined pointwise as $(B\alpha B)_{ij} = B_i \alpha B_j$ for any $i, j \in I_n$.

4. ON COMPACT SOLUTIONS OF EQ. (1.1).

Assume here that $\mathfrak{R} \neq \emptyset$ too. Let $\mathfrak{C}_0 = \mathfrak{C} \cap \mathfrak{R}$ be the set of all compact solutions of Eq. (1.1) and now we need to define the matrix $C \in \mathfrak{F}$ as $C = S \wedge S^2 \wedge \cdots \wedge S^n$ in order to prove that $\mathfrak{R} \neq \emptyset$ iff $C \in \mathfrak{C}_0$. We give some preliminary propositions and lemmas.

PROPOSITION 4.1. If $S_{ij} = 1$ and $S^2_{ij} < 1$ for some $i, j \in I_n$, then we have that $S^2_{ij} = B_j$.

PROOF. Since $S^2_{ij} \geq S_{ij} \wedge S_{jj} \geq B_j$ assume that $S^2_{ij} > B_j$. Then $B_j < S^2_{ij} = S_{ih} \wedge S_{hj} \leq S_{hj}$ for some $h \in I_n$ and this should imply that $S_{hj} = 1$ and hence $S^2_{ij} = S_{ih}$. If $S_{ih} = B_h$, then $A_i \leq B_j < S^2_{ij} = S_{ih} = B_h < A_i$, a contradiction. Thus $S_{ih} = 1$, i.e., $S^2_{ij} = S_{ih} \wedge S_{hj} = 1 \wedge 1 = 1$, a contradiction to the hypothesis that $S^2_{ij} < 1$. Therefore $S^2_{ij} = B_j$.

PROPOSITION 4.2. If $S^2_{ij} < S_{ij}$ for some $i, j \in I_n$, then we have that $S^2_{hj} \leq S_{hj}$ for any $h \in I_n$.

PROOF. Let $S^2_{kj} > S_{kj}$ for some $k \in I_n$. Since $S^2_{kj} = S_{kt} \wedge S_{tj}$ for some $t \in I_n$, we should have that $S^2_{kj} = S_{kt} \wedge S_{tj} > S_{kj}$. Then $t \neq j$ (otherwise $S_{kj} \geq S^2_{kj} > S_{kj}$, a contradiction) and $S_{tj} \geq S^2_{kj} > S_{kj} = B_j$, which should imply that $S_{tj} = 1$. If $S_{kt} = B_t$, then $B_t = S_{kt} \geq S^2_{kj} > S_{kj} = B_j$. If $S_{kt} = 1$, then $B_j < A_k \leq B_t$ and thus $B_t > B_j$ in any case. On the other hand, $S_{ij} > S^2_{ij} \geq B_j$, hence $1 = S_{ij}$, i.e., $S^2_{ij} = B_j$ by Prop. 4.1. But this contradicts the fact that $S^2_{ij} \geq S_{it} \wedge S_{tj} = S_{it} \wedge 1 = S_{it} \geq B_t > B_j$.

PROPOSITION 4.3. Let $k > 1$ be a positive integer. (a) If $S^k_{ij} < S^{k-1}_{ij}$ (resp. (b) If $S^k_{ij} > S^{k-1}_{ij}$), then we have that $S^2_{mj} < S_{mj}$ (resp. $S^2_{im} > S_{im}$) for some $m \in I_n$.

PROOF. We prove the thesis (a) since the thesis (b) can be proved similarly. The thesis (a) is certainly true for $k = 2$ (it suffices to choose $m = i$). Hence assume $k > 2$ and let $m \in I_n$ such that $S^{k-1}_{ij} = S^{k-2}_{im} \wedge S_{mj} > S^k_{ij} \geq S^{k-2}_{im} \wedge S^2_{mj}$. If $S^{k-2}_{im} \wedge S^2_{mj} = S^{k-2}_{im}$, then $S^{k-2}_{im} \geq S^{k-1}_{ij} > S^k_{ij} \geq S^{k-2}_{im}$, a contradiction. Hence $S^{k-2}_{im} \wedge S^2_{mj} = S^2_{mj}$ and therefore $S_{mj} \geq S^{k-1}_{ij} > S^k_{ij} \geq S^2_{mj}$.

If $S^2_{ij} = 1$, the set $L_{ij} = \{t \in I_n : S_{it} = S_{tj} = 1\}$ is certainly nonempty. Then the following results hold:

PROPOSITION 4.4. Let $S_{ij} = S^2_{ij} = 1$ and $S_{tt} < 1$ for any $t \in L_{ij}$.
(a) If $S_{ih} < 1$ (resp. (b) If $S_{hi} < 1$) for some $h \in I_n$, then we have that $S^k_{ih} < 1$ (resp. $S^k_{hj} < 1$) for any integer $k \geq 1$.

PROOF. Let $h \in I_n$ be such that $S_{ih} < 1$ and assume that $S^k_{ih} = 1$ for some integer $k \geq 1$. We could certainly suppose, without loss of generality, $k (\geq 2)$ to be the smallest integer such that $S^k_{ih} = 1 > S^{k-1}_{ih}$. By Prop. 4.3 (b), then $S^2_{im} = 1 > S_{im} = B_m$ for some $m \in I_n$. Let $t \in I_n$ such that $S^2_{im} = S_{it} \wedge S_{tm} = 1$, thus we should have that $A_t \leq B_m < A_i \leq B_t$, i.e., $S_{tt} = 1$. On the other hand, we know that $B_m < A_i \leq B_j$ since $S_{ij} = 1$, thus (since $S_{tm} = 1$) $A_t \leq B_m < B_j$ which should imply that $S_{tj} = 1$, i.e., $t \in L_{ij}$ and therefore the contradiction $1 = S_{tt} < 1$. Thus the thesis (a) is true and similarly one proves the thesis (b).

LEMMA 4.5. If $S^k_{ij} > B_j$ for any integer $k \geq 1$, then we have that $S^k_{ij} = 1$ for any such k.

PROOF. The thesis is true for $k = 1$ and $k = 2$ by Prop. 4.1. Assume $k > 2$ and, reasoning by induction, we must prove that if $S^{k-1}_{ij} = 1$, then $S^k_{ij} = 1$. Indeed, let $S^k_{ij} < 1 = S^{k-1}_{ij}$ and thus $S_{mj} > S^2_{mj}$ for some $m \in I_n$ by Prop. 4.3(a). Moreover, Prop. 4.2 should imply that $S^2_{hj} \leq S_{hj}$ for any $h \in I_n$. Further, we know that

$$B_j < S_{ij}^k = S_{ih_1} \wedge S_{h_1h_2} \wedge \cdots \wedge S_{h_{k-1}j}$$

for some $h_1, h_2 \cdots, h_{k-1} \in I_n(h_o = i, h_k = j)$, hence $S_{h_{k-1}j} = 1$ since $B_j < S_{ij}^k \le S_{h_{k-1}j}$ and $S_{ih_1} = 1$ otherwise $A_i \le B_j < S_{ij}^k \le S_{ih_1} = B_{h_1} < A_i$, a contradiction.

If $A_i \le B_j < S_{ij}^k \le S_{h_{k-2}h_{k-1}} = B_{h_{k-1}}$, then we should deduce that $S_{ih_{k-1}} = 1$ and $S_{h_{k-1}h_{k-1}} = 1$ since $A_{h_{k-1}} \le B_j < B_{h_{k-1}}$.

Thus $S_{h_{k-1}h_{k-1}}^{k-2} = S_{h_{k-1}h_{k-1}} = 1$ and hence we should get the evident contradiction $1 > S_{ij}^k = S_{ih_{k-1}} \wedge S_{h_{k-1}h_{k-1}}^{k-2} \wedge S_{h_{k-1}j} = 1 \wedge 1 \wedge 1 = 1$. Then $S_{h_{k-2}h_{k-1}} = 1$, which should imply that $S_{h_{k-2}j}^2 = S_{h_{k-2}h_{k-1}} \wedge S_{h_{k-1}j} = 1$.

Let

$$\mathcal{K} = \{a \in \{0,1,2,\cdots,k\} : S_{h_ah_{a+1}} = \cdots = S_{h_{k-1}j} = 1 \text{ and } S_{h_aj}^2 = 1\}.$$

This set is nonempty because $k-2 \in \mathcal{K}$. If $b = \min\{a: a \in \mathcal{K}\}$, then $b > 0$ otherwise

$$1 > S_{ij}^k = S_{h_0h_1} \wedge \cdots \wedge S_{h_{k-1}j} = 1,$$

a contradiction. Thus $b \ge 1$, $1 = S_{h_bj} = S_{h_bj}^2$ and $A_i \le B_j < S_{ij}^k \le S_{h_{b-1}h_b}$. If $S_{h_{b-1}h_b} = B_{h_b}$, then $A_i \vee A_{h_b} \le B_j < B_{h_b}$, i.e., $S_{h_bh_b}^{k-2} = S_{h_bh_b} = S_{ih_b} = 1$ and hence

$$1 > S_{ij}^k = S_{ih_b} \wedge S_{h_bh_b}^{k-2} \wedge S_{h_bj} = 1 \wedge 1 \wedge 1 = 1,$$

a contradiction. Thus $S_{h_{b-1}h_b} = 1$ and $S_{h_{b-1}j}^2 = S_{h_{b-1}h_b} \wedge S_{h_bj} = 1 \wedge 1 = 1$, i.e., $b-1 \in \mathcal{K}$, a contradiction to the hypothesis that b is the minimum of $\mathcal{K}$. Therefore $S_{ij}^k = 1$, i.e., the thesis.

LEMMA 4.6. Let $i \ne j$, $S_{ij} = S_{ij}^2 = 1, i, j \notin L_{ij}$ and $S_{tt} < 1$ for any $t \in L_{ij}$. Then we have that $S_{ij}^n < 1$.

PROOF. Let

$$1 = S_{ij}^n = S_{ih_1} \wedge S_{h_1h_2} \wedge \cdots \wedge S_{h_{n-1}j}$$

for some $h_1, h_2, \cdots, h_{n-1} \in I_n(h_0 = i, h_n = j)$. Then we should have that $S_{ih_1} = 1$ and

$$S_{h_1j}^{n-1} = S_{h_1h_2} \wedge \cdots \wedge S_{h_{n-1}j} = 1 \wedge \cdots \wedge 1 = 1,$$

i.e., $S_{h_1j} = 1$ by Prop. 4.4(b), thus $h_1 \in L_{ij}$. Now $S_{ih_2}^2 = S_{ih_1} \wedge S_{h_1h_2} = 1 \wedge 1 = 1$ and $S_{h_2j}^{n-2} = 1$, hence $S_{ih_2} = S_{h_2j} = 1$ by Prop. 4.4 (a), (b) and then we should deduce that $h_2 \in L_{ij}$ too. Now $h_1 \ne h_2$ otherwise $B_{h_1} < A_{h_1} \le B_{h_2} = B_{h_1}$, a contradiction. So continuing, we should get that $h_t \in L_{ij}$ for any $t \in I_{n-1}$ and $h_s \ne h_t$ for any $s, t \in I_{n-1}$ since $B_{h_1} < B_{h_2} < \cdots < B_{h_{n-1}}$. This should imply that card $L_{ij} \ge n-1$, a contradiction to the hypothesis that card $L_{ij} \le n-2$ (since $i, j \notin L_{ij}$). Now we are able to show that

THEOREM 4.7. The matrix $C = \wedge_{k=1}^n S^k$ belongs to $\mathbf{C}$.

PROOF. We must prove that $C_{ij}^2 \ge C_{ij}$ for any $i, j \in I_n$. In order to avoid trivial situations, assume that $i \ne j$. Since

$$C_{ij}^2 = \bigvee_{t=1}^n (C_{it} \wedge C_{tj}) \ge C_{ij} \wedge C_{jj} \ge B_j,$$

the thesis is clearly true if $C_{ij} = B_j$. Let $C_{ij} > B_j$, i.e., $S_{ij}^k > B_j$ for any $k \in I_n$. By Lemma 4.5, we have that $S_{ij}^k = 1$ for any $k \in I_n$ (in particular, $S_{ij} = S_{ij}^2 = S_{ij}^n = 1$) and hence $C_{ij} = 1$. If $i \in L_{ij}$, then we deduce that $C_{ii} = 1$ because $S_{ii}^k = 1$ for any $k \in I_n$ and hence $C_{ij}^2 = C_{ii} \wedge C_{ij} = 1 \wedge 1 = 1$. Similarly one gets $C_{ij}^2 = 1$ if $j \in L_{ij}$. Now assume that $i, j \notin L_{ij}$. By Lemma 4.6, there exists $m \in L_{ij}$ such that

$S_{im} = S_{mj} = S_{mm} = 1$, hence $S^{k-1}_{mm} = 1$ for any integer $k \geq 2$. Thus $S^k_{im} = S_{im} \wedge S^{k-1}_{mm} = 1 \wedge 1 = 1$ and $S^k_{mj} = S^{k-1}_{mm} \wedge S_{mj} = 1 \wedge 1 = 1$ for any $k \in I_n$. Then we have that $C^2_{ij} = C_{im} \wedge C_{mj} = 1 \wedge 1 = 1$, hence we get $C_{ij} = C^2_{ij} = 1$ in any case.

The main result of this section is the following:

THEOREM 4.8. $\Re \neq \emptyset$ iff $C \in \mathfrak{C}_0$. Further, $C \geq R$ for any $R \in \mathfrak{C}_0$.

PROOF. Of course, we are interested in the non-trivial implication. Let $\Re \neq \emptyset$, thus $S \in \Re$ [15] and $I \in \mathfrak{I}$, by Thm. 3.1, with $I \leq S$. We have that $I = I^k \leq S^k$ for any $k \in I_n$, so $I \leq C \leq S$ and then $C \in \mathfrak{C}_0$ by Remark 2.6 and Thm. 4.7. Let $R \in \mathfrak{C}_0$, thus $R \leq S$ and hence $R^k \leq S^k$ for any $k \in I_n$, i.e., $R = R \wedge R^2 \wedge \cdots \wedge R^n \leq C$.

The following example must clarify all the results already established.

EXAMPLE 4.9. Let $n = 3$, $A_1 = B_2 = 0.8$, $A_2 = 0.9$, $A_3 = 0.4$, $B_1 = 0.5$, $B_3 = 0.6$. Then $\Re \neq \emptyset$ since $SoA = B$, where

$$S = \begin{bmatrix} 0.5 & 1 & 0.6 \\ 0.5 & 0.8 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}, \qquad S^2 = \begin{bmatrix} 0.6 & 0.8 & 0.6 \\ 0.6 & 0.8 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}, \qquad S^3 = \begin{bmatrix} 0.6 & 0.8 & 0.6 \\ 0.6 & 0.8 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus

$$W = \begin{bmatrix} 0.5 & 1 & 0.6 \\ 0.5 & 0.8 & 0.6 \\ 0.5 & 1 & 1 \end{bmatrix}, \quad W^2 = W^3 = I = \begin{bmatrix} 0.5 & 0.8 & 0.6 \\ 0.5 & 0.8 & 0.6 \\ 0.5 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0.8 & 0.6 \\ 0.5 & 0.8 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}.$$

Note that S, W, I, C are all distinct between them. Finally, we prove that

THEOREM 4.10. If $B_j > B_t$ for any $t \in I_n - \{j\}$ and $\Re \neq \emptyset$, then we have that $R_{jj} \geq B_j$ for any $R \in \mathfrak{C}_0$.

PROOF. Let $\Re \neq \emptyset$ and $R_{jj} < B_j$ for some $R \in \mathfrak{C}_0$. Let $k \in I_n$ be such that $A_k \wedge R_{kj} = B_j$. Thus $k \neq j$
(otherwise, $R_{jj} \geq B_j$) and then

$$B_j \leq R_{kj} \leq R^2_{kj} = [\bigvee_{\substack{t=1 \\ t \neq j}}^{n} (R_{kt} \wedge R_{tj})] \vee (R_{kj} \wedge R_{jj}).$$

Since $A_k \wedge R_{kt} \leq B_t < B_j \leq A_k$ for any $t \in I_n - \{j\}$, we should deduce that $R_{kt} \leq B_t < B_j$, i.e., $R_{kt} \wedge R_{tj} < B_j$ for any $t \in I_n - \{j\}$ and $R_{kj} \wedge R_{jj} \leq R_{jj} < B_j$. This should imply that $B_j \leq R_{kj} \leq R^2_{kj} < B_j$, a contradiction.

5. ON IDEMPOTENT SOLUTIONS OF EQ. (1.2).

Concerning Eq. (1.2) and following Di Nola [2] (see also [5]), we define the (row-) vectors Q_i, T_i : $\{x_i\} \times X \to L$ by setting $Q_i(x_i, x_j) = Q_{ij}$ and $T_i(x_i, x_k) = T_{ik}$ for any $i, j, k \in I_n$. Thus the study of Eq. (1.2) is equivalent to consider the following system of equations of type (1.1):

$$R \; o \; Q_i = T_i, \quad i = 1, 2, \cdots, n. \tag{5.1}$$

As in [3], let $\Re_i = \Re_i(Q_i, T_i)$ (resp. $\mathfrak{T}_i = \mathfrak{T}(Q_i, T_i) = \Re_i \cap \Re_\wedge$) be the set of all (resp. max- min) transitive solutions of the i-th Eq. (5.1) and, of course, we have that

$$S = \bigcap_{i=1}^{n} \Re_i, \qquad \mathfrak{T}^* = \bigcap_{i=1}^{n} \mathfrak{T}_i,$$

where $\mathfrak{T}^* = \mathfrak{T}^*(Q, T) = S \cap \Re_\wedge$ is the set of all max-min transitive solutions of Eq. (1.2). If $\Re_i \neq \emptyset$, then we can consider [3] the greatest max-min transitive $W_i \in \mathfrak{T}_i$ of the i-th Eq. (5.1) and we define the matrix $W^* = \wedge_{i=1}^{n} W_i$. In [3], it was proved that $\mathfrak{T}^* \neq \emptyset$ iff $W^* \in \mathfrak{T}^*$ and $W^* \geq R$ for any $R \in \mathfrak{T}^*$.

If $\mathfrak{I}_i = \mathfrak{I}_i(Q_i, T_i) = \mathfrak{R}_i \cap \xi$ (resp. $\mathfrak{I}^* = \mathfrak{I}^*(Q,T) = S \cap \xi$) is the set of all idempotent solutions of the i-th Eq. (5.1) (resp. (1.2)), we have obviously that

$$\mathfrak{I}^* = \bigcap_{i=1}^{n} \mathfrak{I}_i \subseteq \mathfrak{T}^*.$$

If $\mathfrak{R}_i \neq \emptyset$, let $I_i = W_i^n$, by Thm. 3.1, be the greatest element of $\mathfrak{I}_i$ for any $i \in I_n$, then we deduce the following result:

THEOREM 5.1. $\mathfrak{I}^* \neq \emptyset$ iff $I^* \in \mathfrak{I}^*$, where $I^* = (\wedge_{i=1}^{n} I_i)^n$. Further, $I^* = (W^*)^n \geq R$ for any $R \in \mathfrak{I}^*$.

PROOF. Let $\mathfrak{I}^* \neq \emptyset$, then $\mathfrak{I}_i \neq \emptyset$ and hence $\mathfrak{R}_i \neq \emptyset$ for any $i \in I_n$. By Thm. 3.1, we can consider $I_i \in \mathfrak{I}_i$ for any $i \in I_n$ and since $I_i \in \xi$ for any $i \in I_n$, we have that I^* belongs to ξ by [6]. Let $R \in \mathfrak{I}^* \subseteq \mathfrak{R}_i$, thus $R \in \mathfrak{I}_i$ for any $i \in I_n$, i.e., $R \leq \wedge_{i=1}^{n} I_i$ and then $R = R^n \leq (\wedge_{i=1}^{n} I_i)^n = I^* \leq I_i^n = I_i$. This means that $I^* \in \mathfrak{R}_i$ for any $i \in I_n$ by Remark 2.6, i.e., $I^* \in S$ and hence I^* lies in $\mathfrak{I}^*$. We note explicitly that $(W^*)^n \leq W_i^n$ for any $i \in I_n$, so that [6]

$$(W^*)^n = (W^*)^{n+1} = [(W^*)^n]^n \leq (\bigwedge_{i=1}^{n} W_i^n)^n = (\bigwedge_{i=1}^{n} I_i)^n = I^*.$$

On the other hand, since $I^* \in \mathfrak{T}^*$, we have that $I^* = (I^*)^n \leq (W^*)^n$ and therefore $(W^*)^n = I^*$.

We can have that $\mathfrak{I}^* \neq \emptyset$ (hence $S \neq \emptyset$), then $\mathfrak{T}^* \neq \emptyset$ but $S \neq \emptyset$ and $\mathfrak{T}^* \neq \emptyset$ do not imply necessarily that $\mathfrak{I}^* \neq \emptyset$ as it is shown in the following:

EXAMPLE 5.2. Let $n = 3$, Q and T be defined as

$$Q = \begin{bmatrix} 1 & 0.6 & 0.4 \\ 0.8 & 0.5 & 0.3 \\ 0.4 & 0.9 & 0.7 \end{bmatrix}, \qquad T = \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.6 & 0.7 & 0.6 \\ 0.5 & 0.6 & 0.6 \end{bmatrix}.$$

Then

$$Q_1 \alpha T_1 = Q_2 \alpha T_2 = \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad Q_3 \alpha T_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0.5 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.6 \end{bmatrix}.$$

Thus $W_3 = Q_3 \alpha T_3$ and

$$W_1 = W_2 = \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.6 & 1 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{hence } S^* = W^* = \begin{bmatrix} 0.6 & 0.7 & 0.6 \\ 0.5 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.6 \end{bmatrix},$$

$$(W^*)^2 = (W^*)^3 = I^* = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.6 \\ 0.5 & 0.6 & 0.6 \end{bmatrix}. \text{ It is easily seen that } S^* o Q = T.$$

Then $S \neq \emptyset$ and $\mathfrak{T}^* \neq \emptyset$, but $(I^* oQ)_{12} = (I^* oQ)_{22} = 0.6 < T_{12} = T_{22} = 0.7$ and hence $\mathfrak{I}^* = \emptyset$.

Note that, as easy examples prove, we can have that $S \neq \emptyset$ and $\mathfrak{T}^* = \emptyset$, hence $\mathfrak{I}^* = \emptyset$. For sake of completeness and in accordance to Remark 3.2, we point out the following result:

THEOREM 5.3. If $\mathfrak{T}^* = \emptyset$, then $W^* = (T^{-1} \alpha T) \wedge S^*$, where $(T^{-1} \alpha T) \in \mathfrak{T}$ is defined pointwise as

$$(T^{-1} \alpha T)_{jk} = \bigwedge_{i=1}^{n} (T_{ij} \alpha T_{ik}) \text{ for any } j,k \in I_n.$$

PROOF. Since $\mathfrak{T}^* \neq \emptyset$, then $W^* \in \mathfrak{T}^* \subseteq \mathbf{S}$ and hence $W^* \leq S^*$ [3]. Since $(W^* o\ W^*) \leq W^*$, we have that $W^* o\ T = W^* o\ (W^* o\ Q) = (W^* o\ W^*)\ o\ Q \leq W^* o\ Q = T$. Thus (cfr. Section 2) $W^* \leq (T^{-1}\alpha T)$, i.e.,
$W^* \leq (T^{-1}\alpha T) \wedge S^*$. If

$$W^*_{jk} < (T^{-1}\alpha T)_{jk} \wedge S^*_{jk} \tag{5.2}$$

for some $j, k \in I_n$, then we should have that $W^*_{jk} = W_h(x_j, x_k) = Q_{jh}\alpha T_{hk}$ for some $h \in I_n$ and (5.2) should imply that

$$Q_{hj}\alpha T_{hk} = W^*_{jk} < S^*_{jk} = Q_{hj}\alpha T_{hk},$$

a contradiction that concludes the proof.

6. ON COMPACT SOLUTIONS OF EQ. (1.2). Let $\mathfrak{C}_{oi} = \mathfrak{C}_{oi}(Q_i, T_i) = \mathfrak{C} \cap \mathfrak{R}_i$ (resp. $\mathfrak{C}^* = \mathfrak{C}^*(Q,T) = \mathfrak{C} \cap \mathbf{S}$) be the set of all compact solutions of the i-th Eq. (5.1) (resp. (1.2)) for any $i \in I_n$. Of course, we have that

$$\mathfrak{C}^* = \bigcap_{i=1}^{n} \mathfrak{C}_{oi}.$$

If $\mathfrak{C}^* \neq \emptyset$, then $\mathbf{S} \neq \emptyset$, hence $\mathfrak{R}_i \neq \emptyset$ for any $i \in I_n$. By Thm. 4.8, we can consider the greatest element $C_i \in \mathfrak{C}_{oi}$ for any $i \in I_n$. Now, if $R \in \mathfrak{C}^* \subseteq \mathfrak{R}_i$, we have that $R \leq C_i$ for any $i \in I_n$, i.e., $R \leq C^* \leq C_i$ for any $i \in I_n$, where $C^* = \wedge_{i=1}^{n} C_i$. Thus $C^* \in \mathfrak{R}_i$ for any $i \in I_n$ by Remark 2.6, i.e., $C^* \in \mathbf{S}$ but generally, C^* does not belong to $\mathfrak{C}^*$ as Example 5.2 shows. Indeed, we have that $C_1 = Q_1\alpha T_1 = C_2 = Q_2\alpha T_2$, $C_3 = Q_3\alpha T_3$ and therefore $C^* = S^* = W^* > (W^*)^2 = (C^*)^2$, i.e., $C^* \notin \mathfrak{C}$ and fortiori, $C^* \notin \mathfrak{C}^*$.

OPEN QUESTION. If $\mathfrak{C}^* \neq \emptyset$, how to characterize, if exists, the greatest element of $\mathfrak{C}^*$?

REMARK 6.1. Example 5.2 proves also that we can have $\mathbf{S} \neq \emptyset$ and $\mathfrak{T} \neq \emptyset$ but $\mathfrak{C}^* = \emptyset$ otherwise, if $\mathfrak{C}^* \neq \emptyset$, should be $\mathfrak{C}_{o2} \neq \emptyset$ and there we should deduce that, for any $R \in \mathfrak{C}^* \subseteq \mathfrak{C}_{02}$, $RoQ_2 = T_2$, i.e., by Thm. 4.9, $0.6 = S^*_{22} \geq R_{22} \geq T_{22} = 0.7$, a contradiction. Therefore $\mathfrak{C}^* = \emptyset$ since $\mathfrak{C}_{02} = \emptyset$.

7. CONCLUDING COMMENTS. Returning to the finite-state system of Sec. 1 represented from Eq. (1.1), applying k times the transition R, we have that $B^{(k)} = R^k oA = R^{k+1} oA = B^{(k+1)}$ if $k = n$ and $R \in \mathfrak{T}$ [6] or if $k = 1$ and $R \in \mathfrak{I}$ or if $k = n-1$ and $R \in \mathfrak{C}_0$ [21], $B^{(k)} = S^k oA = S^{k+2} oA = B^{(k+2)}$ if $k = 3n-4$ or $k = 3n-3$ [11], i.e., concluding as in [3] and [11], the more general the transitivity is requested on R, the slower the speed of convergence of the system is. In [3], the authors characterized also the minimal element of $\mathfrak{T}$ (resp. $\mathfrak{T}^*$) by means of the max-min transitive closures of the minimal element of $\mathfrak{R}$ (resp. $\mathbf{S}$).

OPEN QUESTION: How to characterize, if exist, the minimal elements of $\mathfrak{I}$ (resp. $\mathfrak{I}^*$) and $\mathfrak{C}_0$ (resp. $\mathfrak{C}^*$)?

REFERENCES

1. BIRKHOFF, G. Lattice Theory, 3rd ed., Vol. XXV, AMS Colloquium Publications, Providence, Rhode Island, 1967.

2. DI NOLA, A. Relational equations in totally ordered lattices and their complete resolution, J. Math.Anal. Appl. 107 (1985) 148-155.

3. DI NOLA, A., KOLODZIEJCZYK, W. and SESSA, S. Transitive Solutions of Relational Equations of Finite Sets and Linear Lattices, Lecture Notes in Computer Science (B. Bouchon-Meunier, R.R. Yager and L.A. Zadeh, Eds.), Vol. 521, Springer-Verlag, Berlin (1990), 173-182.

4. DI NOLA, A. and LETTIERI, A. Relation equations in residuated lattices, Rend Circolo Matem. Palermo 38 (1989), 246-256.

5. DI NOLA, A., PEDRYZ, W., SANCHEZ, E. and SESSA, S. Fuzzy Relation Equations and Their Applications to Knowledge Engineering, Kluwer Academic Publishers, Dordrecht, 1989.

6. HASHIMOTO, H. Convergence of powers of fuzzy transitive matrix, Fuzzy Sets and Systems 9 (1983), 153-160.

7. KACPRZYK, J. and FEDRIZZI, M. (Eds.), Combining fuzzy Imprecision with Probabilistic Uncertainty in Decision Making, Vol. 310, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin 1988.

8. KACPRZYK, J. and FEDRIZZI, M. (Eds.), Multiperson Decision Making Models Using Fuzzy Sets and Possibility Theory, Kluwer Academic publishers, Dordrecht 1990.

9. KAUFMANN, A. Introduction to the Theory of Fuzzy Subsets, Vol. I, Academic Press, New York 1975.

10. KIM, K. H. Boolean Matrix Theory and Applications, Marcel Dekker, New York 1982.

11. KOLODZIEJCZYK, W. On transitive solutions of σ-fuzzy relation equations describing fuzzy systems, Internat.J. Gen. Syst. 17 (1990), 277-288.

12. LUCE, R. D. A note on Boolean matrix theory, Proc. Amer. Math. Soc. 3 (1952), 382-388.

13. PEDRYCZ, W. Fuzzy Control and Fuzzy Systems, Research Studies Press Ltd J. Wiley & Sons Inc., Chichester 1989.

14. RUDEANU, S. Boolean Functions and Equations, North-Holland, Amsterdam 1974.

15. SANCHEZ, E. Resolution of composite fuzzy relation equations, Inform. and Control 30 (1976), 38-48.

16. SCHWARTZ, S. On the semigroup of binary relations on a finite set, Czech. Math. J. 20 (1970), 632-679.

17. SHAO, J.Y. and LI, Q. On the indices of convergence of irreducible Boolean matrices, Linear Algebra Appl. 97 (1987), 185-210.

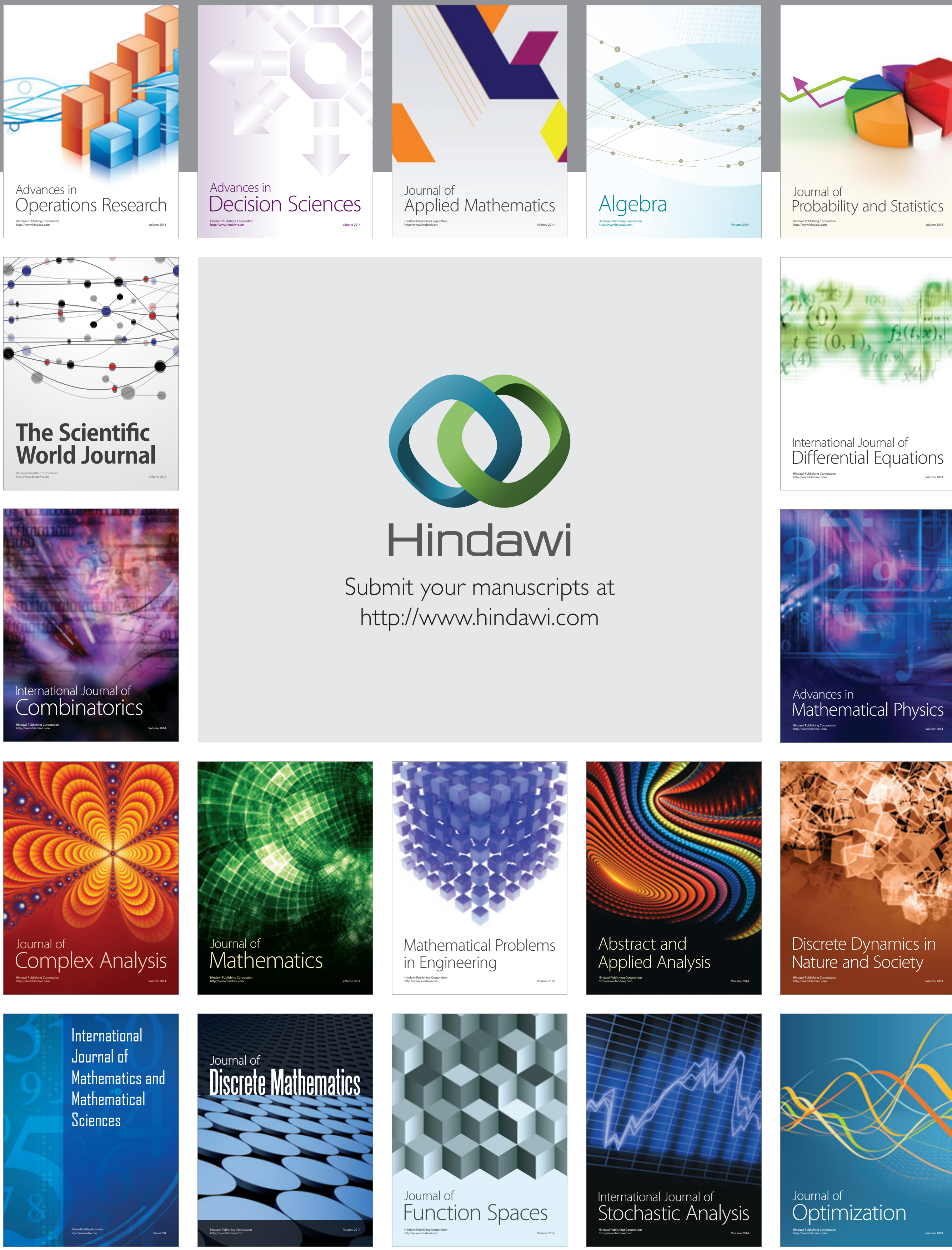
18. SHMUELY, Z. Idempotents in complete posemigroups, Semigroup Forum 17 (1979), 317-324.

19. SHMUELY, Z. The lattice of idempotent binary relations, Algebra Universalis 9 (1979), 297-303.

20. TAHANI, V. A fuzzy model of document retrival systems, Inform. Processing Manag. 12 (1976), 177-187.

21. THOMASON, M. G. Convergence of powers of a fuzzy matrix, J. Math. Anal Appl. 57 (1977), 476-480.

22. ZIMMERMAN, U. Linear and Combinatorial Optimization in Ordered Algebraic Structures, North-Holland, Amsterdam 1981.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization