THE NACHBIN COMPACTIFICATION VIA CONVERGENCE ORDERED SPACES

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ABSTRACT. We construct the Nachbin compactification for a $T_{3.5}$ -ordered topological ordered space by taking a quotient of an ordered convergence space compactification. A variation of this quotient construction leads to a compactification functor on the category of $T_{3.5}$ -ordered convergence ordered spaces.

KEY WORDS AND PHRASES: topological ordered space, convergence ordered space, T_2 -ordered space, $T_{3.5}$ -ordered space, Nachbin compactification.

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0. INTRODUCTION.

The Nachbin (or Stone-Cech-ordered) compactification (see [1], [6]) is the largest T_2 -ordered topological ordered compactification of a $T_{3.5}$ -ordered topological ordered space. In [4], one of the authors and G.D. Richardson constructed an ordered compactification (X^*, φ) for an arbitrary convergence ordered space X . This latter compactification exhibits essentially the same universal property as the Nachbin compactification, but behaves poorly relative to separation properties (see Example 1.4).

Starting with an arbitrary convergence ordered space X , we introduce an equivalence relation $\mathcal{R}$ on the set $|X^*|$ which underlies X^* , and obtain an ordered quotient space $X^*/\mathcal{R}$ which is both compact and T_2 -ordered. We next give two conditions C and O which are necessary and sufficient to make the natural map from X into $X^*/\mathcal{R}$ both an order embedding and a homeomorphic embedding, so that $X^*/\mathcal{R}$ becomes a T_2 -ordered convergence ordered compactification of X . For ordered convergence spaces X satisfying conditions C and O , it turns out that the topological modification λX of X is a $T_{3.5}$ -ordered topological ordered space, and $\lambda(X^*/\mathcal{R})$ is the Nachbin compactification of λX . In particular, if X is assumed to be a $T_{3.5}$ -ordered topological ordered space, then $\lambda(X^*/\mathcal{R})$ is the Nachbin compactification of X .

In addition to giving an alternate construction for the Nachbin compactification, we obtain some interesting results pertaining to convergence ordered compactifications. In Section 3, we define a

regular convergence ordered space satisfying conditions C and O to be a $T_{3.5}$ -ordered convergence ordered space, and we show that for such a space X , the regular modification $r(X^*/\mathcal{R})$ of the quotient $X^*/\mathcal{R}$ is a regular, T_2 -ordered convergence ordered compactification of X . Relative to this compactification functor, the regular, T_2 -ordered, compact convergence spaces (with increasing, continuous maps as morphisms) form an epireflective subcategory of the category of all $T_{3.5}$ -ordered convergence ordered spaces (with increasing, continuous maps as morphisms).

1. PRELIMINARIES.

We introduce some basic notation and terminology and summarize some results from [4]. If $(X, \leq)$ is a poset, and $A \subseteq X$, we denote by $i(A)$, $d(A)$, and $A^\wedge$ the *increasing*, *decreasing*, and *convex* hulls, respectively, of A ; note that $A^\wedge = i(A) \cap d(A)$. Similarly, if $F(X)$ is the set of all (proper) filters on X and $\mathcal{F} \in F(X)$, let $i(\mathcal{F})$, the filter generated by $\{i(F) : F \in \mathcal{F}\}$, be the *increasing hull* of $\mathcal{F}$; the decreasing hull $d(\mathcal{F})$ and convex hull $\hat{\mathcal{F}}$ are defined analogously. A filter $\mathcal{F}$ is said to be *convex* if $\mathcal{F} = \hat{\mathcal{F}}$. Note that $\hat{\mathcal{F}} = i(\mathcal{F}) \vee d(\mathcal{F})$.

If $(X, \leq, \rightarrow)$ is a poset $(X, \leq)$ equipped with a convergence structure $\rightarrow$ which is *locally convex* (i.e., $\hat{\mathcal{F}} \rightarrow x$ whenever $\mathcal{F} \rightarrow x$), then $(X, \leq, \rightarrow)$ is called a *convergence ordered space*; we usually write X rather than $(X, \leq, \rightarrow)$ when there is no danger of ambiguity. A convergence ordered space is T_1 -ordered if the sets $i(x)$ and $d(x)$ are closed for all $x \in X$, and T_2 -ordered if the order $\leq$ is a closed subset of $X \times X$. For any convergence ordered space X , let $CI^*(X)$ (respectively, $CD^*(X)$) denote the set of all continuous, increasing (respectively, decreasing) maps from X into $[0, 1]$.

A convergence ordered space whose convergence structure is a topology is called a *topological ordered space*. Such a space is said to be *convex* if the open monotone (i.e., increasing or decreasing) sets form a subbase for the topology. For the remainder of this paper, we shall adopt the notational abbreviation used in [4] and write "t.o.s." instead of "topological ordered space" and "c.o.s." in place of "convergence ordered space".

A t.o.s. X is said to be $T_{3.5}$ -ordered if it satisfies the following conditions: (1) If $x \in X$, A is a closed subset of X , and $x \notin A$, then there is $f \in CI^*(X)$ and $g \in CD^*(X)$ such that $f(x) = g(x) = 0$ and $f(y) \vee g(y) = 1$, for all $y \in A$; (2) If $x \not\leq y$ in X , there is $f \in CI^*(X)$ such that $f(y) = 0$ and $f(x) = 1$. The $T_{3.5}$ -ordered spaces are precisely those which allow T_2 -ordered t.o.s. compactifications, and all $T_{3.5}$ -ordered spaces are convex.

If X is a $T_{3.5}$ -ordered t.o.s., then the *Nachbin compactification* of X (see [1], [6]) is obtained by embedding X into an "ordered cube", whose component intervals are indexed by $CI^*(X)$. The Nachbin compactification $\beta_0 X$ is characterized by the following well-known result.

PROPOSITION 1.1. If X is a $T_{3.5}$ -ordered t.o.s., then $\beta_0 X$ is T_2 -ordered. Furthermore, if $f : X \rightarrow Y$ is an increasing, continuous map and Y is a compact, T_2 -ordered t.o.s., then f has a unique, increasing, continuous extension $f' : \beta_0 X \rightarrow Y$.

We next describe briefly the construction of the convergence ordered compactification X^* of an arbitrary c.o.s. X described in [4], which has essentially the same lifting property as $\beta_0 X$. Given a c.o.s. X , let X' be the set of all non-convergent maximal convex filters on X , and let $X^* = \{\dot{x} : x \in X\} \cup X'$. Before proceeding further, it will be useful to establish the following proposition about maximal convex filters.

PROPOSITION 1.2. The maximal convex filters on a poset X are precisely the set $\{\hat{\mathcal{F}} : \mathcal{F} \text{ is an ultrafilter on } X\}$.

PROOF. Clearly every maximal convex filter is the convex hull of every finer ultrafilter. Conversely, suppose $\mathcal{F}$ is an ultrafilter on X and $\mathcal{G}$ is a convex filter such that $\hat{\mathcal{F}} \leq \mathcal{G}$. Then for any

convex set $G \in \mathcal{G}$, the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ generated by $\{i(G) \cap F : F \in \mathcal{F}\}$ and $\{d(G) \cap F : F \in \mathcal{F}\}$, respectively, are well-defined filters finer than, and hence equal to, $\mathcal{F}$. Thus $i(G) \in \mathcal{F}$ and $d(G) \in \mathcal{F}$ implies $i(G) \cap d(G) = G \in \mathcal{F}$; therefore $\mathcal{G} = \hat{\mathcal{F}}$. $\blacksquare$

Again assuming that X is an arbitrary c.o.s., let $\varphi : X \rightarrow X^*$ be defined by $\varphi(x) = \dot{x}$, for all $x \in X$. A partial order $\leq^*$ is defined on X^* as follows: $\mathcal{F} \leq^* \mathcal{G}$ iff $i(\mathcal{F}) \leq \mathcal{G}$ (or, equivalently, $d(\mathcal{G}) \leq \mathcal{F}$). Since $x \leq y$ iff $\dot{x} \leq^* \dot{y}$, $\varphi : (X, \leq) \rightarrow (X^*, \leq^*)$ is an order embedding.

If $A \subseteq X$, let $A^* = \{\mathcal{F} \in X^* : A \in \mathcal{F}\}$; if $\mathcal{F} \in \mathbf{F}(X)$, let $\mathcal{F}^*$ denote the filter in $\mathbf{F}(X^*)$ generated by $\{F^* : F \in \mathcal{F}\}$. A convergence structure $\xrightarrow{*}$ on $(X^*, \leq^*)$ is defined as follows: For $\mathcal{A} \in \mathbf{F}(X^*)$,

$$\begin{aligned} \mathcal{A} &\xrightarrow{*} \dot{x} \in \varphi(X) \text{ iff there is } \mathcal{F} \rightarrow x \text{ such that } \mathcal{F}^* \leq \mathcal{A}; \\ \mathcal{A} &\xrightarrow{*} \mathcal{G} \in X^* \text{ iff } \mathcal{G}^* \leq \mathcal{A}. \end{aligned}$$

Writing X^* in place of $(X^*, \leq^*, \xrightarrow{*})$, we state the following result which is proved in [4].

PROPOSITION 1.3. If X is a c.o.s., then (X^*, φ) is a convergence ordered compactification of X . If $f : X \rightarrow Y$ is a continuous, increasing map and Y a compact, regular, T_2 -ordered c.o.s., then f has a unique, increasing, continuous extension $f_* : X^* \rightarrow Y$.

Recall that a convergence space Y is *regular* if $cl_Y \mathcal{F} \rightarrow x$ whenever $\mathcal{F} \rightarrow x$. Here " cl_Y " is the closure operator for Y , and $cl_Y \mathcal{F}$ is the filter on Y generated by $\{cl_Y F : F \in \mathcal{F}\}$.

In [4], a c.o.s. X is defined to be *strongly T_2 -ordered* if X is T_2 (i.e., convergent filters have unique limits) and the following conditions hold: (S_1) if $\mathcal{F} \rightarrow x$, $\mathcal{G} \in X'$, and $i(\mathcal{F}) \leq \mathcal{G}$, then $d(\mathcal{G}) \leq \dot{x}$; (S_2) if $\mathcal{F} \rightarrow x$, $\mathcal{G} \in X'$, and $d(\mathcal{F}) \leq \mathcal{G}$, then $i(\mathcal{G}) \leq \dot{x}$. In Proposition 2.8, [4], it is shown that X^* is T_2 -ordered iff X is strongly T_2 -ordered. As we see in the next example, very nice c.o.s.'s may fail to be strongly T_2 -ordered.

EXAMPLE 1.4. Let X be the Euclidean plane with its usual (product) order and topology. Let $\mathcal{F}$ be the filter on X generated by sets of the form $F_n = \{(a, b) \in X : -\frac{1}{n} < a < 0, b = 0\}$ for each natural number n , and let $x = (0, 0)$. Let $\mathcal{G}$ be the convex hull of any ultrafilter containing the set $S = \{(a, b) \in X : a = -b^{-1}\}$ and coarser than the filter $\mathcal{H}$ generated by sets of the form $H_n = \{(a, b) \in X : b \geq n\}$ for $n = 1, 2, 3, \dots$. Note that (S_1) is violated by $\mathcal{F}, \mathcal{G}$ and x ; thus the compactification X^* of X is not T_2 -ordered. $\blacksquare$

2. $\beta_0 X$ AS A QUOTIENT OF X^* .

Let $(X, \leq, \rightarrow)$ be any c.o.s., and let (X^*, φ) be the convergence ordered compactification of X constructed in the last section. By Proposition 1.3 there is, for any $f \in CI^*(X)$, a unique, continuous, increasing extension $f_* : X^* \rightarrow [0, 1]$.

We define an equivalence relation $\mathcal{R}$ on X^* as follows: $\mathcal{R} = \{(\mathcal{F}, \mathcal{G}) \in X^* \times X^* : f_*(\mathcal{F}) = f_*(\mathcal{G}), \text{ for all } f \in CI^*(X)\}$. Let σ be the projection map of X^* onto $X^*/\mathcal{R}$ (i.e., for each $\mathcal{F} \in X^*$, $\sigma(\mathcal{F}) = [\mathcal{F}]$, where $[\mathcal{F}]$ is the $\mathcal{R}$ -equivalence class containing $\mathcal{F}$). A partial order $\leq_{\mathcal{R}}$ on $X^*/\mathcal{R}$ is defined as follows:

$$[\mathcal{F}] \leq_{\mathcal{R}} [\mathcal{G}] \text{ iff } f_*(\mathcal{F}) \leq f_*(\mathcal{G}) \text{ in } R \text{ for all } f \in CI^*(X).$$

We also impose on $X^*/\mathcal{R}$ the quotient convergence structure which is described (see [2]) as follows: If $\Phi \in \mathbf{F}(X^*/\mathcal{R})$ and $[\mathcal{F}] \in X^*/\mathcal{R}$, then $\Phi \rightarrow [\mathcal{F}]$ in $X^*/\mathcal{R}$ iff there is $\mathcal{F}' \in [\mathcal{F}]$ and there is a filter $\mathcal{A} \in \mathbf{F}(X^*)$ such that $\mathcal{A} \xrightarrow{*} \mathcal{F}'$ in X^* and $\sigma(\mathcal{A}) \leq \Phi$.

THEOREM 2.1. For any c.o.s. X , $X^*/\mathcal{R}$ is a compact, T_2 -ordered c.o.s.

PROOF. $X^*/\mathcal{R}$ is obviously compact. To show that $X^*/\mathcal{R}$ is T_2 -ordered, it is sufficient (by Proposition 1.2, [4]) to show that if $\Phi, \Theta \in \mathcal{F}(X^*/\mathcal{R})$, $\Phi \rightarrow [\mathcal{F}]$ and $\Theta \rightarrow [\mathcal{G}]$ in $X^*/\mathcal{R}$, and $\Phi \times \Theta$ has a trace on the order $\leq_{\mathcal{R}}$, then $[\mathcal{F}] \leq_{\mathcal{R}} [\mathcal{G}]$.

If $f \in CI^*(X)$, define $f_{\mathcal{R}} : X^*/\mathcal{R} \rightarrow [0, 1]$ by $f_{\mathcal{R}}([\mathcal{F}]) = f_*(\mathcal{F})$, for all $\mathcal{F} \in X^*$. It is easy to verify that $f_{\mathcal{R}}$ is well-defined and $f_{\mathcal{R}} \in CI^*(X^*/\mathcal{R})$. If $\Phi \rightarrow [\mathcal{F}]$ and $\Theta \rightarrow [\mathcal{G}]$ in $X^*/\mathcal{R}$ and $\Phi \times \Theta$ has a trace on $\leq_{\mathcal{R}}$, it follows that $f_{\mathcal{R}}(\Phi) \times f_{\mathcal{R}}(\Theta)$ has a trace on the order of $[0, 1]$; since $[0, 1]$ is T_2 -ordered, $f_{\mathcal{R}}([\mathcal{F}]) = f_*(\mathcal{F}) \leq f_*(\mathcal{G}) = f_{\mathcal{R}}([\mathcal{G}])$. The latter inequality holds for all $f \in CI^*(X)$, and so $[\mathcal{F}] \leq_{\mathcal{R}} [\mathcal{G}]$, which establishes that $X^*/\mathcal{R}$ is T_2 -ordered. $\blacksquare$

For an arbitrary c.o.s. X , we have already defined the continuous, increasing maps $\varphi : X \rightarrow X^*$ and $\sigma : X^* \rightarrow X^*/\mathcal{R}$; we define $\varphi_{\mathcal{R}} : X \rightarrow X^*/\mathcal{R}$ by $\varphi_{\mathcal{R}} = \sigma \circ \varphi$. It is clear that $\varphi_{\mathcal{R}}(X)$ is dense in the compact, T_2 -ordered c.o.s. $X^*/\mathcal{R}$. We are now interested in characterizing those spaces X for which $(X^*/\mathcal{R}, \varphi_{\mathcal{R}})$ is a compactification. With this goal in mind, we introduce the following conditions.

CONDITION C. For any maximal convex filter $\mathcal{F}$ on X , $\mathcal{F} \rightarrow x$ in X iff $f(\mathcal{F}) \rightarrow f(x)$ in $[0, 1]$ for all $f \in CI^*(X)$.

CONDITION O. For any points x, y in X , $x \leq y$ in X iff $f(x) \leq f(y)$ in $[0, 1]$, for all $f \in CI^*(X)$.

It is easy to verify that any $T_{3.5}$ -ordered t.o.s. satisfies Conditions C and O.

LEMMA 2.2. If X is a c.o.s. satisfying Conditions C and O, then $[\dot{x}] = \{\dot{x}\}$, for all $x \in X$.

PROOF. $CI^*(X)$ separates points in X by Condition O, and so σ is one-to-one on $\varphi(X)$. This implies $\dot{y} \notin [\dot{x}]$ if $x \neq y$. Next, assume that there is $\mathcal{F} \in X' \cap [\dot{x}]$. Then $f_*(\mathcal{F}) = f_*(\dot{x}) = f(x)$ for all $f \in CI^*(X)$; in other words, $f(\mathcal{F}) \rightarrow f(x)$ in $\mathcal{R}$, for all $f \in CI^*(X)$. Condition C then implies $\mathcal{F} \rightarrow x$ in X , contradicting the assumption $\mathcal{F} \in X'$. $\blacksquare$

THEOREM 2.3. Let X be a c.o.s. Then $\varphi_{\mathcal{R}} : X \rightarrow X^*/\mathcal{R}$ is an order and a homeomorphic embedding iff X satisfies Conditions C and O.

PROOF. Suppose that X satisfies Conditions C and O. Then $\varphi_{\mathcal{R}}$ is one-to-one since $CI^*(X)$ separates points in X . Also note that $\varphi_{\mathcal{R}} = \sigma \circ \varphi = (\sigma|_{\varphi(X)}) \circ \varphi$, and thus $\sigma|_{\varphi(X)}$ is one-to-one.

Let $\Phi \rightarrow [\dot{x}]$ in $X^*/\mathcal{R}$. Then there is $\mathcal{A} \in \mathcal{F}(X^*)$ such that $\mathcal{A} \xrightarrow{*} \dot{x}$ in X^* and $\Phi \geq \sigma(\mathcal{A})$. By definition of $*$ convergence in X^* , there is a filter $\mathcal{F}$ on X such that $\mathcal{F} \rightarrow x$ and $\mathcal{A} \geq \mathcal{F}^*$. Therefore, $\varphi_{\mathcal{R}}^{-1}(\Phi) \geq \varphi_{\mathcal{R}}^{-1}(\sigma(\mathcal{A})) \geq \varphi_{\mathcal{R}}^{-1}(\sigma(\mathcal{F}^*)) = \varphi^{-1} \cdot (\sigma|_{\varphi(X)})^{-1}(\sigma(\mathcal{F}^*))$. It follows by Lemma 2.2 that $(\sigma|_{\varphi(X)})^{-1}(\sigma(\mathcal{F}^*)) \geq \mathcal{F}^*$. Consequently, $\varphi_{\mathcal{R}}^{-1}(\Phi) \geq \varphi^{-1}(\mathcal{F}^*) = \mathcal{F} \rightarrow x = \varphi_{\mathcal{R}}^{-1}([\dot{x}])$, i.e. $\varphi_{\mathcal{R}}^{-1}(\Phi) \rightarrow \varphi_{\mathcal{R}}^{-1}([\dot{x}])$. Thus $\varphi_{\mathcal{R}}^{-1}$ is continuous.

Let $[\dot{x}] \leq_{\mathcal{R}} [\dot{y}]$ in $X^*/\mathcal{R}$; then for any $f \in CI^*(X)$, $f_*(\dot{x}) \leq f_*(\dot{y})$, i.e. $f_*(\varphi(x)) \leq f_*(\varphi(y))$, which implies $f(x) \leq f(y)$, for all $f \in CI^*(X)$. By Condition O, $x \leq y$. Thus $\varphi_{\mathcal{R}}^{-1}$ is increasing, and we conclude that $\varphi_{\mathcal{R}}$ is an order and homeomorphic embedding.

Conversely, assume that $\varphi_{\mathcal{R}}$ is both an order and homeomorphic embedding. Let $\mathcal{F}$ be a maximal convex filter on X such that, for some $x \in X$, $f(\mathcal{F}) \rightarrow f(x)$ for all $f \in CI^*(X)$. Suppose $\mathcal{F} \rightarrow x$ is not true. Then we need to consider two cases.

CASE 1. $\mathcal{F} \rightarrow y$ and $y \neq x$. This implies that for each $f \in CI^*(X)$, $f(\mathcal{F}) \rightarrow f(y)$. From this we deduce that $[\dot{x}] = [\dot{y}]$, which is a contradiction, since $\varphi_{\mathcal{R}}$ is assumed to be one-to-one.

CASE 2. $\mathcal{F} \in X'$. This leads to the conclusion that $[\mathcal{F}] = [\dot{x}]$; in other words, $\varphi_{\mathcal{R}}(\mathcal{F}) \rightarrow [\dot{x}]$ in $X^*/\mathcal{R}$, which implies $\mathcal{F} \rightarrow x$ in X , since $\varphi_{\mathcal{R}}$ is a homeomorphic embedding. This contradicts $\mathcal{F} \in X'$. We therefore conclude that X satisfies Condition C.

Finally, let $x, y \in X$ such that $f(x) \leq f(y)$ for all $f \in CI^*(X)$. Then $f_*(\varphi(x)) \leq f_*(\varphi(y))$ for all $f \in CI^*(X)$, i.e. $f_*(\dot{x}) \leq f_*(\dot{y})$ for all $f \in CI^*(X)$. This implies $[\dot{x}] \leq_{\mathcal{R}} [\dot{y}]$ in $X^*/\mathcal{R}$, and $x \leq y$

follows since φ_R is an order embedding. Therefore, X satisfies Condition O. I

THEOREM 2.4. For every c.o.s. X satisfying Conditions C and O, $((X^*/\mathcal{R}), \varphi_R)$ is a T_2 -ordered c.o.s. compactification of X . Furthermore, for any compact, regular, T_2 -ordered c.o.s. Y and for any continuous, increasing map $f : X \rightarrow Y$, there is a unique, continuous, increasing extension $f_R : X^*/\mathcal{R} \rightarrow Y$.

PROOF. The first assertion is an immediate corollary of Theorem 2.3. The second follows easily with the help of Proposition 1.3. I

For any c.o.s. X , let $\omega_o X$ be the t.o.s. consisting of the poset $(X, \leq)$ with the weak topology induced by $CI^*(X)$. Note that $CI^*(X) = CI^*(\omega_o X)$.

PROPOSITION 2.5. Let X be a c.o.s. satisfying Condition C. Let $i : X \rightarrow \omega_o X$ be the identity map. Then i is an order isomorphism and a homeomorphism relative to ultrafilter convergence.

PROOF. It is obvious that i is a continuous order isomorphism. Let $\mathcal{F} \rightarrow x$ in $\omega_o X$, where $\mathcal{F}$ is an ultrafilter. By Proposition 1.2, $\hat{\mathcal{F}}$ is a maximal convex filter and $f(\mathcal{F}) \rightarrow f(x)$ implies $f(\hat{\mathcal{F}}) \rightarrow f(x)$ in $[0, 1]$, for all $f \in CI^*(X)$. Condition C thus guarantees that $\hat{\mathcal{F}} \rightarrow x$ in X , and hence $\mathcal{F} \rightarrow x$ in X . I

PROPOSITION 2.6. If X is a c.o.s. satisfying Conditions C and O, then $\omega_o X$ is a $T_{3.5}$ -ordered t.o.s.

PROOF. First observe that $\omega_o X$ also satisfies Condition C and O; O is obvious, and C follows from Proposition 2.5, since X and $\omega_o X$ have the same ultrafilter convergence and hence, by Proposition 1.2, the same convergence of maximal convex filters.

For $f \in CI^*(\omega_o X)$, let I be the closed interval $[0, 1]$ indexed by f , and let $P = \Pi\{I_f : f \in CI^*(X)\}$ be equipped with the usual product order and product topology. Then P is a compact, T_2 -ordered t.o.s. Define $\varphi_o : \omega_o X \rightarrow P$ by $\varphi_o(x) = \hat{x}$, where $\hat{x} : CI^*(\omega_o X) \rightarrow [0, 1]$ is given by $\hat{x}(f) = f(x)$, for all $f \in CI^*(\omega_o X)$. Since $\omega_o X$ has the weak topology induced by $CI^*(\omega_o X) = CI^*(X)$, and $CI^*(\omega_o X)$ separates points in $\omega_o X$ by Condition O, φ_o is a topological embedding (see 8.12, [10]). By Condition O, φ_o is also an order embedding. I

Given a c.o.s. X satisfying C and O, we introduce some additional functional notation. Let e_o be the evaluation embedding of the $T_{3.5}$ -ordered t.o.s. $\omega_o X$ into its Nachbin compactification $\beta_o X$, and let $e = e_o \cdot i : X \rightarrow \beta_o(\omega_o X)$. The unique extension of e to X^* (guaranteed by Proposition 1.3) is denoted by e_* , and the extension of e to $X^*/\mathcal{R}$ (guaranteed by Theorem 2.4) is denoted by e_R . If $f \in CI^*(X) = CI^*(\omega_o X)$, the unique extensions of f in $CI^*(X^*)$ and $CI^*(\beta_o(\omega_o X))$ (see Proposition 1.3 and 2.4) are denoted by f_* and f^* , respectively. The following commutative diagram is helpful in keeping track of these various maps.

$$\begin{array}{ccccc}
 X & \xrightarrow{\varphi_o} & X^* & \xrightarrow{\sigma} & X^*/\mathcal{R} \\
 i \downarrow & \searrow & \downarrow e_* & & \swarrow e_R \\
 \omega_o X & \rightarrow & \beta_o(\omega_o X) & & \\
 & e_o & & &
 \end{array}$$

THEOREM 2.7. If X is any c.o.s. satisfying C and O, then e_R is an order isomorphism and a homeomorphism relative to ultrafilter convergence.

PROOF. Since $[\mathcal{F}] = [\mathcal{G}]$ in $X^*/\mathcal{R}$ iff $e_*(\mathcal{F}) = e_*(\mathcal{G})$ iff $e_R([\mathcal{F}]) = e_R([\mathcal{G}])$, it follows that e_R is one-to-one. Furthermore, $e(X)$ is dense in $\beta_o(\omega_o X)$, which implies that the extension e_R is onto $\beta_o(\omega_o X)$. It follows from Theorem 2.4 that e_R is continuous and increasing. Finally, if $\mathcal{H}$ is an ultrafilter on $\beta_o(\omega_o X)$ and $\mathcal{H} \rightarrow a$ in $\beta_o(\omega_o X)$, then there is $\alpha \in X^*/\mathcal{R}$ such that $e_R^{-1}(\mathcal{H}) \rightarrow \alpha$ in

$X^*/\mathcal{R}$ since the latter space is compact. It follows by uniqueness of filter limits in both spaces and the continuity of $e_{\mathcal{R}}$ that $e_{\mathcal{R}}^{-1}(a) = \alpha$. $\blacksquare$

If X is any convergence space, let λX denote its *topological modification* (i.e., λX is the set $|X|$ equipped with the finest topological structure coarser than X .) If X is a c.o.s. satisfying C and O , we obtain from Proposition 2.5 and Theorem 2.7 that $\lambda X = \omega_o X$ and $\lambda(X^*/\mathcal{R})$ is a compact, T_2 -ordered t.o.s. homeomorphic and order isomorphic under $e_{\mathcal{R}}$ to $\beta_o(\omega_o X)$. Let $\varphi_o : \omega_o X \rightarrow X^*/\mathcal{R}$ be defined by $\varphi_o = \sigma \circ \varphi \circ i^{-1} = \varphi_{\mathcal{R}} \circ i^{-1}$.

COROLLARY 2.8. If X is a c.o.s. satisfying C and O , then $(\lambda(X^*/\mathcal{R}), \varphi_o)$ is the Nachbin compactification of $\omega_o X = \lambda X$. If X is a $T_{3.5}$ -ordered t.o.s., then $(\lambda(X^*/\mathcal{R}), \varphi_o)$ is the Nachbin compactification of X .

One question which deserves clarification is the status of $X^*/\mathcal{R}$ as a "quotient" of X^* . We have indeed equipped $X^*/\mathcal{R}$ with the quotient convergence structure, but can we interpret $\leq_{\mathcal{R}}$ as the "quotient order" relative to the order $\leq^*$ defined on X^* ? Various notions of "quotient order" have been considered (for instance, see [5] and [8]), but the order $\leq_{\mathcal{R}}$ is generally different than these. Instead of regarding the order and convergence structures of $X^*/\mathcal{R}$ separately, we think that it is appropriate to consider the notion of a "quotient c.o.s.", where order and convergence structures are considered together. From this perspective, the next theorem indicates that $X^*/\mathcal{R}$ is indeed a quotient c.o.s. of X^* , at least in the category of c.o.s.'s which satisfy Conditions C and O .

THEOREM 2.10. For a c.o.s. X , let X^* and $X^*/\mathcal{R}$ be defined as before. Let Y be any c.o.s. satisfying C and O , and let $h : X^*/\mathcal{R} \rightarrow Y$. Then h is continuous and increasing iff $h \circ \sigma : X^* \rightarrow Y$ is continuous and increasing.

PROOF. If h is continuous and increasing, the same is obviously true for $h \circ \sigma$.

Conversely, suppose $h \circ \sigma$ is continuous and increasing. Let $\Phi \rightarrow [\mathcal{F}]$ in $X^*/\mathcal{R}$; then there is $\mathcal{F}' \in [\mathcal{F}]$ and a filter $\mathcal{A}$ on X^* such that $\mathcal{A} \rightarrow \mathcal{F}'$ in X^* and $\Phi \geq \sigma(\mathcal{A})$. Hence $h \circ \sigma(\mathcal{A}) \rightarrow h \circ \sigma(\mathcal{F}')$ in Y , by continuity of $h \circ \sigma$. But $\Phi \geq \sigma(\mathcal{A})$ and $\sigma(\mathcal{F}') = [\mathcal{F}]$, so $h(\Phi) \rightarrow h([\mathcal{F}])$, implying that h is continuous.

To show that h is increasing, let e_Y be the natural map from Y into $\beta_o(\omega_o Y)$ and consider $g = e_Y \circ h \circ \sigma \circ \varphi : X \rightarrow \beta_o(\omega_o Y)$. Since $g : \omega_o X \rightarrow \beta_o(\omega_o Y)$ is also continuous and increasing, there is a continuous, increasing extension $g^* : \beta_o(\omega_o X) \rightarrow \beta_o(\omega_o Y)$ which makes the diagram below commute.

$$\begin{array}{ccccccc}
 X & \xrightarrow{\varphi} & X^* & \xrightarrow{\sigma} & X^*/\mathcal{R} & \xrightarrow{h} & Y \\
 e \searrow & & & & \swarrow e_{\mathcal{R}} & & \downarrow e_Y \\
 & & \beta_o(\omega_o X) & & \xrightarrow{g^*} & & \beta_o(\omega_o Y)
 \end{array}$$

Thus $e_Y \circ h \circ \sigma \circ \varphi = g^* \circ e_{\mathcal{R}} \circ \sigma \circ \varphi$, and since $\sigma \circ \varphi : X \rightarrow X^*/\mathcal{R}$ is a dense injection, $e_Y \circ h = g^* \circ e_{\mathcal{R}}$. But e_Y is an order embedding, so $h = e_Y^{-1} \circ g^* \circ e_{\mathcal{R}}$, and h is increasing. $\blacksquare$

3. $T_{3.5}$ -ORDERED CONVERGENCE ORDERED SPACES.

In this brief concluding section, we introduce the notion of a $T_{3.5}$ -ordered c.o.s., describe the largest regular, T_2 -ordered c.o.s. compactification of such a space, and interpret this compactification in the language of category theory. The necessary categorical terminology can be found in [7].

In [3], a convergence space X is defined to be *completely regular* if it allows a symmetric com-

pactification. In [9], it is shown that the Hausdorff, completely regular convergence spaces, which we shall refer to as $T_{3,5}$ convergence spaces, are precisely those convergence spaces which allow a regular, Hausdorff convergence space compactification.

Given a convergence space X , let rX denote the *regular modification* of X (i.e., rX is the set $|X|$ equipped with the finest regular convergence structure coarser than the original convergence structure on X).

We define a c.o.s. X which is regular and satisfies conditions C and O to be a $T_{3,5}$ -ordered c.o.s.. It follows by Proposition 2.5 that a $T_{3,5}$ -ordered c.o.s. X has the same ultrafilter convergence as its topological modification $\lambda X = \omega_o X$.

THEOREM 3.1. Let X be a $T_{3,5}$ -ordered c.o.s. and let $\eta_o X = r(X^*/\mathcal{R})$ be the regular modification of $X^*/\mathcal{R}$. Then $(\eta_o X, \varphi_{\mathcal{R}})$ is a regular, T_2 -ordered c.o.s. compactification of X . If Y is a regular, T_2 -ordered, compact c.o.s. and $f : X \rightarrow Y$ is continuous and increasing, then f has a unique, continuous, increasing extension $f_* : \eta_o X \rightarrow Y$.

PROOF. By Theorem 2.3, $\varphi_{\mathcal{R}} : X \rightarrow X^*/\mathcal{R}$ is an order embedding and a homeomorphic embedding. By the functorial properties of the regular modification and the fact that $rX = X$, it follows that $\varphi_{\mathcal{R}} : X \rightarrow \eta_o X$ is continuous. Because $X^*/\mathcal{R}$ and $\eta_o X$ have the same ultrafilter convergence, it is easy to verify that the regular modification of $\varphi_{\mathcal{R}}(X)$ (considered as a subspace of $X^*/\mathcal{R}$) coincides with $\varphi_{\mathcal{R}}(X)$ considered as a subspace of $\eta_o X$. From this we see that $\varphi_{\mathcal{R}}^{-1}$ is also continuous, and the first assertion is established. The second assertion is an immediate consequence of Theorem 2.4. I

We denote by $\mathcal{C}$ the category of all $T_{3,5}$ ordered c.o.s.'s, with increasing continuous maps as morphisms; let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ consisting of all regular, compact, T_2 -ordered c.o.s.'s. If $\iota : \mathcal{D} \rightarrow \mathcal{C}$ is the inclusion functor, it follows by Theorem 3.1 that the functor $\eta_o : \mathcal{C} \rightarrow \mathcal{D}$, which assigns to each object X in $\mathcal{C}$ its compactification $\eta_o X$ and to each morphism $f : X \rightarrow Y$ in $\mathcal{C}$ the extension $f_* : \eta_o X \rightarrow \eta_o Y$ whose existence follows by Theorem 3.1, is the left adjoint of ι .

THEOREM 3.2. If $\mathcal{C}$ and $\mathcal{D}$ are the categories defined in the preceding paragraph, then $\mathcal{D}$ is an epireflective subcategory of $\mathcal{C}$.

If X is a $T_{3,5}$ -ordered t.o.s., it is generally not true that $\beta_o X = \eta_o X$, although it is true in this case that $\beta_o X = \lambda(\eta_o X)$.

The $T_{3,5}$ convergence spaces mentioned earlier in this section are the $T_{3,5}$ -ordered c.o.s.'s for which the partial order is equality. Indeed, any $T_{3,5}$ convergence space X , equipped with the trivial order (equality), satisfies Condition C and O relative to $CI^*(X) = C^*(X)$, the set of all continuous maps from X into $[0, 1]$. For such a space X , $\eta_o X$ (which also has the trivial order) coincides with the largest regular, Hausdorff convergence space compactification of X constructed in [9].

REFERENCES

- [1] P. Fletcher and W. Lindgren, *Quasi-Uniform Spaces*, Lect. Notes in Pure and Appl. Math., Vol. 77, Marcel Dekker, Inc., New York (1982).
- [2] D.C. Kent, "Convergence Quotient Maps", *Fund. Math.* **65** (1969) 197-205.
- [3] D.C. Kent and G.D. Richardson, "Completely Regular and ω -Regular Spaces", *Proc. Amer. Math. Soc.* **82** (1981), 649-652.
- [4] _____, "A Compactification for Convergence Ordered Spaces", *Canad. Math. Bull.* **27** (1984) 505-512.

- [5] S.D. McCartan, "A Quotient Ordered Space", *Proc. Camb. Phil. Soc.* **64** (1968), 317-322.
- [6] L. Nachbin, *Topology and Order*, Van Nostrand, *New York Math. Studies*, **4**, Princeton, N.J. (1965).
- [7] G. Preuss, *Theory of Topological Structures*, D. Reidel Publ. Co., Dordrecht (1987).
- [8] H.A. Priestley, "Ordered Topological Spaces and the Representation of Distributive Lattices", *Proc. London Math. Soc.* (3) **24** (1972), 507-530.
- [9] G.D. Richardson and D.C. Kent, "Regular Compactification of Convergence Spaces", *Proc. Amer. Math. Soc.* **31** (1972), 571-573.
- [10] S. Willard, *General Topology*, Addison-Wesley Publ. Co., Reading, Mass. (1970).