THE FREE A-RING IS A GRADED A-RING

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ABSTRACT. In this paper, we define the free A -ring over K on a set X , categorically, and parallel some results from the theory of free algebras. We show that the free A -ring over K on X , denoted by $A_K\{X\}$, is graded.

KEY WORDS AND PHRASES. A -ring, free A -ring.

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DEFINITION 1. Let A be a K -algebra. An **A -ring over K** is a K -algebra B together with a K -algebra homomorphism $f_B: A \rightarrow B$. An **A -ring homomorphism** between A -rings B and C is a K -algebra homomorphism $g: B \rightarrow C$ such that $gf_B = f_C$.

DEFINITION 2. Let X be a set. The **free A -ring**, $A_K\{X\}$, over K on X is an A -ring containing X such that for every A -ring B and function $f: X \rightarrow B$ there is a unique A -ring homomorphism $f_*: A_K\{X\} \rightarrow B$ that extends f . $A_K\{X\}$ is the free object on X in the category of A -rings over K .

Just from the definitions of the terms involved it is easy to see that $A_K\{X\} = A_K^* K \langle X \rangle$, the coproduct of A and $K \langle X \rangle$, the free K -algebra on X . Also $A_K\{X, Y\} = (A_K\{X\})_K\{Y\}$. To see this recall that $K \langle X, Y \rangle = K \langle X \rangle *_K K \langle Y \rangle$ so that $A_K\{X, Y\} \simeq A_K^* K \langle X, Y \rangle \simeq A_K^* (K \langle X \rangle *_K K \langle Y \rangle) \simeq (A_K^* K \langle X \rangle)_K K \langle Y \rangle \simeq (A_K\{X\})_K\{Y\}$. Also using a categorical argument with just the definitions of the terms involved it is easy to prove that $A_K\{X, Y\} \simeq A_K\{X\}_A^* A_K\{Y\}$.

In Cartan and Eilenberg [8, p. 146] the term free A -ring on a set X is used. The terminology may be some what misleading since this notion of a free A -ring is not the free object on X in the category of A -rings. The free A -ring discussed there is essentially $A \otimes_{\mathbb{Z}} K \langle X \rangle$.

Let us call an A -ring that is graded as a K -algebra a **graded A -ring** in case A is homogeneous of degree 0, i.e., the image of A is contained in the homogeneous component of degree 0.

We define the **tensor A -ring of an A -bimodule M over K** to be the A -bimodule $T_A(M) = A \oplus M \oplus (M \otimes_A M) \oplus \cdots \oplus (\otimes^n M) \oplus \cdots$. $T_A(M)$ is made into a ring in the same fashion as the tensor algebra. $T_A(M)$ is a K -algebra since A is a K -algebra and M is an A -bimodule over K . A is a sub- K -algebra of $T_A(M)$ and thus $T_A(M)$ is canonically an A -ring. Also M is a sub- A -bimodule. $T_A(M)$ is by construction a graded A -ring. As before we adopt the notation $A = \otimes^0 M$ and $M_n = \otimes^n M$. As in the case of the tensor algebra, $T_A(M)$ satisfies the following universal mapping property:

THEOREM 1. For all A -rings B and A -bimodule homomorphism $f: M \rightarrow B$, there is a unique A -ring homomorphism $f_*: T_A(M) \rightarrow B$ that extends f .

PROOF. Since B is an A -ring we have the canonical K -algebra homomorphism from A to B . Denote this map by f_o . Denote f by f_1 . Let $f_n: \otimes_A^n M \rightarrow B$ be that unique A -bimodule homomorphism such that $f_n(m_1 \otimes m_2 \otimes \cdots \otimes m_n) = f_1(m_1)f(m_2) \cdots f_n(m_n)$. The f_n 's induce an A -bimodule homomorphism $f_*: T_A(M) \rightarrow B$ by $f_*(\sum p) = \sum f_p(z_p)$. Note that $f_{r+s}(z_r w_s) = f_r(z_r)f_s(w_s)$ since if $z_r = \sum_i m_{i1} \otimes \cdots \otimes m_{ir}$ and $w_s = \sum_j m'_{j1} \otimes \cdots \otimes m'_{js}$ then

$$\begin{aligned} f_{r+s}(z_r w_s) &= f_{r+s}(\sum_{i,j} m_{i1} \otimes \cdots \otimes m_{ir} \otimes m'_{j1} \otimes \cdots \otimes m'_{js}) \\ &= \sum_{i,j} f_1(m_{i1}) \cdots f_1(m_{ir}) f_1(m'_{j1}) \cdots f_1(m'_{js}) \\ &= f_r(\sum_i m_{i1} \otimes \cdots \otimes m_{ir}) f_s(\sum_j m'_{j1} \otimes \cdots \otimes m'_{js}) \end{aligned}$$

We now show that f_* is a ring homomorphism. $f_*(\sum_p z_p \sum_q w_q)$

$$\begin{aligned} &= f_*(\sum_{p,q} z_p w_q) = f_*(z_o w_o + (z_o w_o + (z_o w_1 + z_1 w_o) + \cdots)) \\ &= f_o(z_o w_o) + f_1(z_o w_1 + z_1 w_o) + \cdots \\ &= f_o(z_o)f_o(w_o) + f_o(z_o)f_1(w_1) + f_1(z_1)f_o(w_o) + \cdots \\ &= (f_o(z_o) + \cdots)(f_o(w_o) + f_1(w_1) + \cdots) \\ &= f_*(\sum_p z_p) (f_*(\sum_q w_q)). \end{aligned}$$

$f_*|_A = f_o$ so f_* is an A -ring homomorphism and $f_*|_M = f_1$. If $g: T_A(M) \rightarrow B$ is another A -ring homomorphism such that $g|_M = f_1$ then since $m_1 \otimes \cdots \otimes m_n \in M_n$ is $m_1 m_2 \cdots m_n, m_i \in M$ then $g(m_1 \otimes \cdots \otimes m_n) = g(m_1 \cdots m_n) = g(m_1) \cdots g(m_n) = f_1(m_1) \cdots f_1(m_n) = f_n(m_1 \otimes \cdots \otimes m_n)$ so that since f_n is unique with respect to this property we have $g|_{M_n} = f_n$ for each $n = 0, 1, \dots$, but this means $g = f_*$.

The free K -algebra is an example of a graded K -algebra. The following theorem shows that the free A -ring is a graded A -ring.

THEOREM 2. $A_K\{X\}$ is the tensor A -ring $T_A(F_X)$, where F_X is the free A -bimodule over K on X .

PROOF. Consider the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{i} & F_X & \xrightarrow{j} & T_A(F_X) \\ & \searrow \emptyset & \downarrow f & \swarrow f_* & \\ & & B & & \end{array}$$

where i, j are inclusion maps, B is an arbitrary A -ring and $\emptyset$ is any function from X to B . Since B is an A -ring then B is an A -bimodule over K so there is a unique A -bimodule homomorphism $f: F_X \rightarrow B$ that extends $\emptyset$. By the universal property of $T_A(F_X)$ given by Theorem 1 we have a unique A -ring homomorphism $f_*: T_A(F_X) \rightarrow B$ that extends f . Thus f_* extends $\emptyset$ and if $g: T_A(F_X) \rightarrow B$ is any other A -ring homomorphism that extends $\emptyset$ then $g|_{F_X}$ is an A -bimodule homomorphism from F_X to B that extends $\emptyset$ so that $g|_{F_X} = f$. But f_* is unique with respect to extending f so we must have $g = f_*$.

Because of this theorem $A_K\{X\}$ could also be called the **polynomial K -algebra over A in the noncommuting indeterminates X** , where the scalars $a \in A$ do not necessarily commute with the indeterminates.

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