ON REGULAR AND SIGMA-SMOOTH TWO VALUED MEASURES AND LATTICE GENERATED TOPOLOGIES

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ABSTRACT. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X . $I(\mathcal{L})$ denotes the non-trivial zero one valued finitely additive measures on $A(\mathcal{L})$, the algebra generated by $\mathcal{L}$, and $IR(\mathcal{L})$ those elements of $I(\mathcal{L})$ that are $\mathcal{L}$ -regular. It is known that $I(\mathcal{L})=IR(\mathcal{L})$ if and only if $\mathcal{L}$ is an algebra. We first give several new proofs of this fact and a number of characterizations of this in topological terms.

Next we consider, $I(\sigma^*, \mathcal{L})$ the elements of $I(\mathcal{L})$ that are σ -smooth on $\mathcal{L}$, and $IR(\sigma, \mathcal{L})$ those elements of $I(\sigma^*, \mathcal{L})$ that are $\mathcal{L}$ -regular. We then obtain necessary and sufficient conditions for $I(\sigma^*, \mathcal{L})=IR(\sigma, \mathcal{L})$, and in particular, we obtain conditions in terms of topological demands on associated Wallman spaces of the lattice.

KEY WORDS AND PHRASES. Regular and sigma smooth two valued measures, normal lattices, regular lattices, T_2 lattices, countably paracompact and countably bounded, separation and semi-separation of lattices, pre-measures, I -lattice, etc.

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1. INTRODUCTION

In this paper we wish to determine when certain classes of measures are equal, and to obtain necessary and sufficient conditions for such equality to hold, emphasizing topological characterizations.

To be specific let X be an abstract set, $\mathcal{L}$ a lattice of subsets of X . Let $A(\mathcal{L})$ denote the algebra generated by the lattice $\mathcal{L}$, and $I(\mathcal{L})$ the collection of non-trivial zero-one valued finitely additive measures on $A(\mathcal{L})$. $IR(\mathcal{L})$ will denote measures in $I(\mathcal{L})$ that are $\mathcal{L}$ -regular on $A(\mathcal{L})$, i.e. if $\mu \in IR(\mathcal{L})$ and $B \in A(\mathcal{L})$ then there exists a $L \in \mathcal{L}$ st $B \supseteq L$ and $\mu(B) = \mu(L)$. $I(\sigma^*, \mathcal{L})$ will denote those elements of $I(\mathcal{L})$ that are sigma-smooth on $\mathcal{L}$, i.e. if $L_n \in \mathcal{L}$ $n=1, 2, \dots, \infty$ and $L_n \downarrow \emptyset$ then for $\mu \in I(\sigma^*, \mathcal{L})$, $\lim \mu(L_n) = 0$. $IR(\sigma, \mathcal{L})$ will denote those measures in $I(\sigma^*, \mathcal{L})$ that are $\mathcal{L}$ -regular.

The first area of concern is when $I(\mathbf{L})=IR(\mathbf{L})$. It is well known that this is true iff $\mathbf{L}$ is an algebra. We give several proofs of this, highlighting topological considerations, to be more precise $I(\mathbf{L})=IR(\mathbf{L})$ is equivalent to the following:

- a) The lattice $V(\mathbf{L})$ (see below for definitions) in the space $I(\mathbf{L})$ is regular.
- b) The topology of closed sets $\tau V(\mathbf{L})$ in $I(\mathbf{L})$ is T_1 .
- c) The lattice of sets $V(\mathbf{L})$ in $I(\mathbf{L})$ is disjunctive.

The second main area of concern is determining conditions for $I(\sigma^*, \mathbf{L})=IR(\sigma, \mathbf{L})$, and conversely what this implies for the lattice. We show (see below for definitions) that $I(\sigma^*, \mathbf{L})=IR(\sigma, \mathbf{L})$ is equivalent to: The lattice $V(\sigma, \mathbf{L})$ in the space $I(\sigma^*, \mathbf{L})$ is regular. We also show that if $\mathbf{L}$ is disjunctive and $W(\sigma, \mathbf{L})$ is prime complete or $\mathbf{L}$ is normal and countably compact and if $I(\sigma^*, \mathbf{L}), V(\sigma, \mathbf{L})$ is T_1 then $I(\sigma^*, \mathbf{L})=IR(\sigma, \mathbf{L})$. Also suppose $\tau \mathbf{L} \supseteq E \supseteq A(\mathbf{L}) \supseteq \mathbf{L}$ then if $\tau \mathbf{L}$ is $\mathbf{L}$ cb or more generally if E (and thus $A(\mathbf{L})$) is $\mathbf{L}$ cb and a) Either $S(\mathbf{L}) \supseteq \sigma(\mathbf{L})$ (Where $S(\mathbf{L})$ are the lattice Souslin derived sets. In particular if $\rho(\mathbf{L})=\sigma(\mathbf{L})$.) and $\mathbf{L}$ is delta or b) If $\mathbf{L}$ is complement generated (and not necessarily delta) then $IR(\sigma, \mathbf{L})=I(\sigma^*, \mathbf{L})$.

2. BACKGROUND AND NOTATION

We begin by reviewing some notation and terminology which is fairly standard (see Alexandroff [1], Frolik [4], and Szeto [7]). We supply background material for the readers convenience.

Let X be abstract set, and $\mathbf{L}$ a lattice of subsets of X st $X, \emptyset \in \mathbf{L}$. A delta lattice is one that is closed under countable intersections, and the delta lattice generated by $\mathbf{L}$ is denoted $\delta(\mathbf{L})$. In addition $\mathbf{L}$ is complement generated iff for every element $L \in \mathbf{L}$ there exists a sequence of subsets $A_i \in \mathbf{L}$ $i=1, 2, \dots$ and $L = \bigcap A_i$ (where $'$ denotes complement). $\mathbf{L}$ is countably paracompact if for every sequence $L_n \in \mathbf{L}$ and $L_n \downarrow \emptyset$ there exists $B_n \in \mathbf{L}$ st $B_n \downarrow \emptyset$ and $B_n \supseteq L_n$ for every n . A tau lattice is one that is closed under arbitrary intersections, and the tau lattice generated by $\mathbf{L}$ is denoted by $\tau \mathbf{L}$. $A(\mathbf{L})$ will denote the algebra generated by the lattice $\mathbf{L}$.

Let $\mathbf{L}_1, \mathbf{L}_2$ be two lattices st $\mathbf{L}_2 \supseteq \mathbf{L}_1$, then $\mathbf{L}_1$ semi-separates (ss) $\mathbf{L}_2$ if for $L_1 \in \mathbf{L}_1, L_2 \in \mathbf{L}_2$ and $L_1 \cap L_2 = \emptyset$, then there exists an $A_1 \in \mathbf{L}_1$, $A_1 \supseteq L_2$ such that $A_1 \cap L_1 = \emptyset$.

Let $I(\mathbf{L})$ denote the set of non-trivial two valued $\{0, 1\}$ finitely additive measures on the algebra generated by $\mathbf{L}$, and let $I(\sigma^*, \mathbf{L})$ denote those elements of $I(\mathbf{L})$ that are sigma-smooth on $\mathbf{L}$, i.e. if $\{L_n\} \in \mathbf{L}$, $L_n \downarrow \emptyset$ and $\mu \in I(\sigma^*, \mathbf{L})$, $\lim \mu(L_n) = 0$ as $n \rightarrow \infty$. $I(\sigma, \mathbf{L})$ denotes those elements of $I(\mathbf{L})$ that are sigma-smooth on $A(\mathbf{L})$, i.e. if $\{A_n\} \in A(\mathbf{L})$, $A_n \downarrow \emptyset$, and $\mu \in I(\sigma, \mathbf{L})$ $\lim \mu(A_n) = 0$ as $n \rightarrow \infty$. This is equivalent to countable additivity on $A(\mathbf{L})$. $IR(\mathbf{L})$ will stand for the measures on $A(\mathbf{L})$ that are $\mathbf{L}$ -regular on $A(\mathbf{L})$, i.e. $\mu \in IR(\mathbf{L})$ $\mu(A) = \sup \{ \mu(L) \mid L \in \mathbf{L}, L \supseteq A \}$ and $A \in A(\mathbf{L})$. This is equivalent to being $\mathbf{L}$ -regular on $\mathbf{L}$. $IR(\sigma, \mathbf{L})$ denotes the set of $\mu \in IR(\mathbf{L})$ that are σ -smooth on $\mathbf{L}$. The obvious relations hold, $I(\mathbf{L}) \supseteq I(\sigma^*, \mathbf{L}) \supseteq I(\sigma, \mathbf{L}) \supseteq IR(\sigma, \mathbf{L})$ and $I(\mathbf{L}) \supseteq IR(\mathbf{L})$. The support of a measure $S(\mu), \mu \in I(\mathbf{L})$ is defined as $S(\mu) = \bigcap \{ L \in \mathbf{L} \mid \mu(L) = 1 \}$.

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two lattices of sets of X st $\mathbf{L}_2 \supseteq \mathbf{L}_1$ then $\mathbf{L}_2$ is $\mathbf{L}_1$ countably bounded (cb) if for $L_2 \in \mathbf{L}_2$ and $L_2 \downarrow \emptyset$, then there exists $L_1 \in \mathbf{L}_1, L_1 \downarrow \emptyset$ and $L_1 \supseteq L_2$.

A lattice is said to be disjunctive, if for any $x \in X$ and $L \in \mathbf{L}$ such that $x \notin L$ there exists $L' \in \mathbf{L}$ such that $x \in L'$ and $L \cap L' = \emptyset$. $\mathbf{L}$ is said to be regular if for $x \in X$ $x \notin L$ $L \in \mathbf{L}$ then there exists $L_1, L_2 \in \mathbf{L}$ $x \in L_1$, $L_2 \supseteq L$ and $L_1 \cap L_2 = \emptyset$. $\mathbf{L}$ is said to be normal if for $L_1, L_2 \in \mathbf{L}$ and $L_1 \cap L_2 = \emptyset$, there exist $L_3, L_4 \in \mathbf{L}$ st $L_3 \supseteq L_1$, $L_4 \supseteq L_2$ and $L_3 \cap L_4 = \emptyset$. $\mathbf{L}$ is said to be countably compact if for any $\{L_n\} \in \mathbf{L}$ and $\bigcap L_n = \emptyset$ $n=1, 2, \dots$, then there exists a finite subindexing st $\bigcap L_{n_i} = \emptyset$ $n_i=1$ to N . A lattice is said to be T_1 if for $x, y \in X$ there exist $L_1, L_2 \in \mathbf{L}$ st $x \in L_1$, $y \notin L_1$ and $y \in L_2$, $x \notin L_2$.

$\rho(\mathbf{L})$ denotes the smallest collection of sets that is closed under countable unions and intersections and contains $\mathbf{L}$. $\sigma(\mathbf{L})$ will stand for the smallest σ -algebra containing $\mathbf{L}$. $S(\mathbf{L})$ will stand for the collection of Souslin sets generated by $\mathbf{L}$.

Note: For $\mu, \mu_1 \in I(\mathbf{L})$, we write $\mu \leq \mu_1 (\mathbf{L})$ if $\mu(L) \leq \mu_1(L)$ for all $L \in \mathbf{L}$.

We now note some measure equivalences of topological properties: 1) $\mathbf{L}$ is disjunctive iff for all $x \in X$ $\mu_x \in IR(\sigma, \mathbf{L})$ where μ_x is the point mass measure, i.e. $\mu_x(A) = 1$ if $x \in A$, $\mu_x(A) = 0$ if $x \notin A$, $A \in \mathbf{L}$. 2) $\mathbf{L}$ is regular iff $\mu \leq \mu_1 (\mathbf{L})$ where $\mu, \mu_1 \in I(\mathbf{L})$ implies $S(\mu) = S(\mu_1)$. 3) $\mathbf{L}$ is normal iff $\mu \in I(\mathbf{L})$ and $\mu_1, \mu_2 \in IR(\mathbf{L})$ implies that if $\mu \leq \mu_1 (\mathbf{L})$ and $\mu \leq \mu_2 (\mathbf{L})$ then $\mu_1 = \mu_2$. 4) $\mathbf{L}$ is countably compact iff $\mu \in I(\mathbf{L})$ implies that $\mu \in I(\sigma^*, \mathbf{L})$. The proofs are not difficult and thus we will only prove the third result for the sake of completeness. Further facts about regular and normal lattices appear in Eid [3] and Grassi [5].

Theorem 2.1: $\mathbf{L}$ is normal iff $\mu \in I(\mathbf{L})$ $\mu_1, \mu_2 \in IR(\mathbf{L})$ and $\mu \leq \mu_1 (\mathbf{L}), \mu \leq \mu_2 (\mathbf{L})$ implies $\mu_1 = \mu_2$.

Proof: Let $\mathbf{L}$ be normal and let $\mu \in I(\mathbf{L})$ $\mu \leq \mu_1 (\mathbf{L}), \mu \leq \mu_2 (\mathbf{L})$ $\mu_1, \mu_2 \in IR(\mathbf{L})$, assume $\mu_1 \neq \mu_2$. Then there exists a $L \in \mathbf{L}$ st $\mu_1(L) = 1$ $\mu_2(L) = 0$ say, and since $\mu_2 \in IR(\mathbf{L})$ and $\mu_2(L) = 1$ there exists $L' \in \mathbf{L}$ st $\mu_2(L') = 1$ and $L' \supseteq L^c$, thus $L \cap L' = \emptyset$. But $\mathbf{L}$ is normal thus there exists $L_1, L_2 \in \mathbf{L}$ st $L_1 \supseteq L$ $L_2 \supseteq L'$ and $L_1 \cap L_2 = \emptyset$, which implies that $\mu_1(L_1) = 1$ $\mu_2(L_2) = 1$ or $\mu_1(L_1) = 0$ $\mu_2(L_2) = 0$ and thus $\mu(L_1) = \mu(L_2) = 0$. Also $L_1 \cup L_2 = X$, and therefore $\mu(X) = \mu(L_1) + \mu(L_2) - \mu(L_1 \cap L_2) = 0$, a contradiction. Therefore $\mu_1 = \mu_2$ and the condition holds.

Note: A fact we will use in the second part of the proof and in the proceeding parts of the paper is that there exists a one-one correspondence between prime $\mathbf{L}$ -filters and elements of $I(\mathbf{L})$, and a one-one correspondence between $\mathbf{L}$ -ultrafilters and elements of $IR(\mathbf{L})$. This correspondence is set up by letting $\mu \in I(\mathbf{L})$ and $H = \{ L \in \mathbf{L} \mid \mu(L) = 1 \}$. Then H is a prime $\mathbf{L}$ -filter and conversely if H is a prime $\mathbf{L}$ -filter there exists a measure $\mu \in I(\mathbf{L})$ associated with H st if $L \in H$, $\mu(L) = 1$. A similar correspondence holds for H and $\mu \in IR(\mathbf{L})$ in which case H is an $\mathbf{L}$ -ultrafilter.

Now we return to the proof of the theorem, conversely, let $\mu \in I(\mathbf{L})$ and $\mu \leq \mu_1 (\mathbf{L}), \mu_1 \leq \mu_2 (\mathbf{L})$ for $\mu_1, \mu_2 \in IR(\mathbf{L})$ imply $\mu_1 = \mu_2$, and assume $\mathbf{L}$ is not normal. Then there exists $L_1, L_1^c \in \mathbf{L}$ st $L_1 \cap L_1^c = \emptyset$ and $H = \{ L \in \mathbf{L} \mid L \supseteq L_1 \text{ or } L \supseteq L_1^c \}$ has the finite intersection property and thus there exists an associated measure $\mu \in I(\mathbf{L}')$ associated with the filter base H st $\mu(L') = 1$ $L \in H$. Now let $L_2 \in \mathbf{L}$ and suppose that $\mu(L_2) = 0$, then L_2 does not contain L_1 thus $L_1 \cap L_2 \neq \emptyset$. Since the collection $\{ L_1 \cap L_2 \mid \mu(L_2) = 1 \}$ has the fip thus there exists a $\mu_1 \in IR(\mathbf{L})$ st $\mu_1(L_1) = 1$ and $\mu \leq \mu_1 (\mathbf{L})$. By similar reasoning there exists a $\mu_2 \in IR(\mathbf{L})$ st $\mu \leq \mu_2 (\mathbf{L})$ $\mu_2(L_1^c) = 1$. By hypothesis $\mu_1 = \mu_2$. Hence $\mu_1(L_1) = \mu_1(L_1^c) = 1$. Therefore $\mu_1(L_1 \cap L_1^c) = 1$. But $L_1 \cap L_1^c = \emptyset$ thus $\mu_1(L_1 \cap L_1^c) = 0$, a contradiction. Thus $\mathbf{L}$ must be normal.

We now prove a result that will be useful in the sequel.

Theorem 2.2: Let $\mathbf{L}$ be normal and countably paracompact, then if $\mu \in I(\sigma^*, \mathbf{L})$ there exists a unique $\mu_1 \in IR(\sigma, \mathbf{L})$ st $\mu \leq \mu_1 (\mathbf{L})$.

Proof: Let $\mu \in I(\sigma^*, \mathbf{L})$ and $\mu_1 \in IR(\mathbf{L})$ st $\mu \leq \mu_1 (\mathbf{L})$. Then we must prove $\mu_1 \in I(\sigma, \mathbf{L})$. Let $A_n \in \mathbf{L}$ $A_n \downarrow \emptyset$. Since $\mathbf{L}$ is countably paracompact there exists $\{B_n\} \downarrow \emptyset$, $B_n \in \mathbf{L}$ and $B_n' \supseteq A_n$ for every n . Since $B_n' \supseteq A_n$ and $\mathbf{L}$ is normal and $A_n \cap B_n = \emptyset$, there exists $C_n, D_n \in \mathbf{L}$ st $C_n \supseteq A_n$ and $D_n \supseteq B_n$ st $D_n \cap C_n = \emptyset$. Then $B_n \supseteq D_n \supseteq C_n \supseteq A_n$ and we can assume without loss of generality that these inclusions hold with $D_n \downarrow \emptyset$. Then $\mu_1(A_n) \leq \mu_1(C_n) \leq \mu(C_n) \leq \mu(D_n)$ and since $B_n \downarrow \emptyset$ $D_n \downarrow \emptyset$ plus the fact $\mu \in I(\sigma^*, \mathbf{L})$ imply that $\lim \mu(D_n) = 0$ as $n \rightarrow \infty$. Then $\lim \mu_1(A_n) = 0$ as $n \rightarrow \infty$, and $\mu_1 \in IR(\sigma, \mathbf{L})$. Uniqueness follows from normality.

Next we consider various sets of measures defined on the algebra generated by a lattice $\mathbf{L}$. For example consider $I(\mathbf{L}), I(\sigma^*, \mathbf{L}), IR(\mathbf{L})$, or $IR(\sigma, \mathbf{L})$. Denote such sets by I . Also consider the collection of sets $\mathcal{H}(\mathbf{L})$ where $\mathcal{H}(\mathbf{L}) = \{ \mathcal{H}(L) : L \in \mathbf{L} \}$ and $\mathcal{H}(L) = \{ \mu \in I : \mu(L) = 1 \}$. Then the following hold: a) $\mathcal{H}(A \cup B) = \mathcal{H}(A) \cup \mathcal{H}(B)$ for $A, B \in \mathbf{L}$. b) $\mathcal{H}(A) \cap \mathcal{H}(B) = \mathcal{H}(A \cap B)$ $A, B \in \mathbf{L}$. c) $\mathcal{H}(A)' = \mathcal{H}(A')$ for $A \in \mathbf{L}$. d) If $A \supseteq B$ then $\mathcal{H}(A) \supseteq \mathcal{H}(B)$ $A, B \in \mathbf{L}$. e) If $\mathbf{L}$ is disjunctive (if necessary) and $\mathcal{H}(A) \supseteq \mathcal{H}(B)$ then $A \supseteq B$ $A, B \in \mathbf{L}$. f) The collection $\mathcal{H}(\mathbf{L})$ is a lattice and $\mathcal{H}(A(\mathbf{L})) = A(\mathcal{H}(\mathbf{L}))$.

We will assume in discussing $\mathcal{H}(\mathbf{L})$ for convenience, that $\mathbf{L}$ is disjunctive, although it will be clear that this assumption is not always needed.

If $\mu \in I(\mathbf{L})$ then define a measure on $A(\mathcal{H}(\mathbf{L}))$ by $\mu^\wedge \in I(\mathcal{H}(\mathbf{L}))$ by $\mu^\wedge(\mathcal{H}(A)) = \mu(A)$ for $A \in A(\mathbf{L})$. Conversely if $\mu^\wedge \in I(\mathcal{H}(\mathbf{L}))$ define a measure on $A(\mathbf{L})$ by $\mu(A) = \mu^\wedge(\mathcal{H}(A))$ for $A \in A(\mathbf{L})$. Then the following hold:

Theorem 2.3: If $\mathbf{L}$ is disjunctive (if necessary) then there is a 1-1 correspondence between the sets $I(\mathbf{L})$ and $I(\mathcal{H}(\mathbf{L}))$ given by $\mu \leftrightarrow \mu^\wedge$. Further $\mu \in I(\mathbf{L})$ is σ -smooth or regular iff $\mu^\wedge \in I(\mathcal{H}(\mathbf{L}))$ is σ -smooth or $\mathcal{H}(\mathbf{L})$ regular.

If $I = I(\mathbf{L})$ we let $\mathcal{H}(\mathbf{L}) = V(\mathbf{L})$.

If $I = I(\sigma^*, \mathbf{L})$ we let $\mathcal{H}(\mathbf{L}) = V(\sigma, \mathbf{L})$.

If $I = IR(\mathbf{L})$ we let $\mathcal{H}(\mathbf{L}) = W(\mathbf{L})$.

If $I = IR(\sigma, \mathbf{L})$ we let $\mathcal{H}(\mathbf{L}) = W(\sigma, \mathbf{L})$.

These sets are topologized by taking $\mathcal{H}(\mathbf{L})$, $\mathcal{H}(\mathbf{L}) \in \mathcal{H}(\mathbf{L})$ as a basis for the closed sets, and will be referred to as generalized Wallman spaces

3 THE SPACES $IR(\mathbf{L})$ AND $I(\mathbf{L})$

In this section we investigate a variety of conditions which are equivalent to $IR(\mathbf{L}) = I(\mathbf{L})$ both abstractly and from a topological point of view with respect to the space $I(\mathbf{L})$, $\tau V(\mathbf{L})$. This will be useful for our subsequent analysis of $I(\sigma^*, \mathbf{L})$, as well as being interesting in its own right.

Theorem 3.1: Let $\mathbf{L}$ be a lattice of subsets of X , then the following are equivalent:

- $IR(\mathbf{L}) = I(\mathbf{L})$
- $V(\mathbf{L})$ in the space $I(\mathbf{L})$ is regular
- The topology of closed sets $\tau V(\mathbf{L})$ in $I(\mathbf{L})$ is T_1
- The lattice of sets $V(\mathbf{L})$ in $I(\mathbf{L})$ is disjunctive
- $\mathbf{L}$ is an algebra.

Proof: We show first that $IR(\mathbf{L}) = I(\mathbf{L})$ iff $IR(\mathbf{L}') = IR(\mathbf{L})$.

If $IR(\mathbf{L}) = I(\mathbf{L})$ and if $\mu \in IR(\mathbf{L}')$, then $\mu \in I(\mathbf{L})$ and thus $\mu \in IR(\mathbf{L})$. Also if $\mu \in IR(\mathbf{L})$ then $\mu \in I(\mathbf{L}')$ and there exists $\mu_1 \in IR(\mathbf{L}')$ st $\mu \leq \mu_1$ ($\mathbf{L}'$). But $\mu_1 \in I(\mathbf{L}) = IR(\mathbf{L})$ and therefore $\mu = \mu_1 \in IR(\mathbf{L}')$. Conversely if $IR(\mathbf{L}') = IR(\mathbf{L})$ let $\mu \in I(\mathbf{L})$, then there exists $\mu_1 \in IR(\mathbf{L})$ st $\mu \leq \mu_1$ ($\mathbf{L}$) or $\mu_1 \leq \mu$ ($\mathbf{L}'$). But $\mu_1 \in IR(\mathbf{L}')$, thus $\mu_1 = \mu \in IR(\mathbf{L}') = IR(\mathbf{L})$ and $I(\mathbf{L}) = IR(\mathbf{L})$.

Next we wish to show $IR(\mathbf{L}) = I(\mathbf{L})$ iff the lattice $V(\mathbf{L})$ in the space $I(\mathbf{L})$ is regular.

Let $IR(\mathbf{L}) = I(\mathbf{L})$ assume that $V(\mathbf{L})$ is not regular then there exists $V(\mathbf{L}) \in V(\mathbf{L})$ $\mu \in I(\mathbf{L})$ st $\mu \notin V(\mathbf{L})$ $H = \{ V(\mathbf{L}') \mid V(\mathbf{L}') \supseteq V(\mathbf{L}) \text{ or } \mu \in V(\mathbf{L}') \}$ has the finite intersection property and thus there exists a $\mu_1^\wedge \in I(V(\mathbf{L}'))$ st $\mu_1^\wedge(V(\mathbf{L}')) = 1$ $V(\mathbf{L}') \in H$ and $\mu^\wedge \leq \mu_1^\wedge(V(\mathbf{L}'))$. Projecting down $\mu \leq \mu_1$ ($\mathbf{L}'$), and since $IR(\mathbf{L}) = I(\mathbf{L})$ $\mu = \mu_1$. Then projecting upward $\mu_1^\wedge = \mu^\wedge$, $\mu^\wedge(V(\mathbf{L}')) = \mu_1^\wedge(V(\mathbf{L}')) = 1$ and $\mu^\wedge(V(\mathbf{L})) = \mu_1^\wedge(V(\mathbf{L})) = 1$, a contradiction.

Conversely let $V(\mathbf{L})$ be regular in $I(\mathbf{L})$ let $\mu \in I(\mathbf{L})$, $L \in \mathbf{L}$, and $\mu(L') = 1$. Therefore $\mu \notin V(\mathbf{L})$. Since $V(\mathbf{L})$ is regular there exists $V(\mathbf{L}_1')$, $V(\mathbf{L}_2') \in V(\mathbf{L})$ st $V(\mathbf{L}_1') \cap V(\mathbf{L}_2') = \emptyset$ and $\mu \in V(\mathbf{L}_1')$ and $V(\mathbf{L}_2') \supseteq V(\mathbf{L})$. But this implies that $\mu \in V(\mathbf{L}_2')$ and $L' \supseteq L_2$ $\mu(L_2) = 1$ and $\mu \in IR(\mathbf{L})$, therefore $IR(\mathbf{L}) \supseteq I(\mathbf{L})$ and $I(\mathbf{L}) = IR(\mathbf{L})$.

Next we show $IR(\mathbf{L}) = I(\mathbf{L})$ iff the topology of closed sets $\tau V(\mathbf{L})$ in $I(\mathbf{L})$ is T_1 .

$\tau V(\mathbf{L})$ is T_1 iff $V(\mathbf{L})$ is T_1 . Assume that $IR(\mathbf{L}) = I(\mathbf{L})$ let $\mu_1, \mu_2 \in I(\mathbf{L}) = IR(\mathbf{L})$ and $\mu_1 \neq \mu_2$. Then there exists $L_1, L_2 \in \mathbf{L}$ st $\mu_1(L_1) = 1$, $\mu_2(L_1) = 0$, $\mu_1(L_2) = 0$, and $\mu_2(L_2) = 1$, which implies that $\mu_1 \in V(\mathbf{L}_2')$, $\mu_2 \notin V(\mathbf{L}_2')$, $\mu_1 \notin V(\mathbf{L}_1')$, $\mu_2 \in V(\mathbf{L}_1')$, i.e. $V(\mathbf{L})$ is T_1 .

Conversely let $V(L)$ be T_1 and let $\mu \in I(L)$. If $\mu \notin IR(L)$ then there exists $v \in IR(L)$ and $L_1 \in L$ such that $\mu \in V(L_1')$, $v \in V(L_1')$ and $\mu \leq v$ (L). Since $V(L)$ is T_1 , there exists $L_2 \in L$ such that $\mu \in V(L_2')$ and $v \in V(L_2')$, i.e. $\mu(L_2)=1$ and $v(L_2)=0$, a contradiction.

Next we show $IR(L)=I(L)$ iff the lattice of sets $V(L)$ in $I(L)$ is disjunctive.

Assume that $IR(L)=I(L)$. Let $\mu \in I(L)$ and suppose $L \in L$ $\mu \notin V(L)$. Since $IR(L)=I(L)$ there exists $L_1 \in L$ st $L' \supseteq L_1$ $\mu(L_1)=1$ and $\mu \in V(L_1)$ and $V(L) \cap V(L_1)=\emptyset$, thus $V(L)$ is disjunctive.

Conversely let $V(L)$ be disjunctive, let $\mu \in I(L)$, let $L \in L$, and let $\mu(L)=1$ (and hence $\mu \notin V(L')$). Since $V(L)$ is disjunctive there exists a $V(L_1) \in V(L)$ st $\mu \in V(L_1)$ and $V(L_1) \cap V(L)=\emptyset$. But this implies that $L' \supseteq L_1$ $\mu(L_1)=1$, and $\mu \in IR(L)$. Therefore $IR(L)=I(L)$.

Finally, now we claim $I(L)=IR(L)$ iff L is an algebra, i.e. $L=L'$.

Let L be an algebra and $\mu \in I(L)$ then since $L=L'$ μ is trivially regular and $IR(L)=I(L)$. Conversely let $I(L)=IR(L)$ and assume that $L \neq L'$, i.e. that L is not an algebra. Thus there exists a $L' \in L$ st $L' \notin L$ and look at $H=\{L \mid L \supseteq L' \text{ or } L \supseteq L''\}$. Then H has the fip and thus there exists a $\mu \in I(L)$ st $\mu(L)=1$, $L \in H$. For $\mu(L_1')=1$ $L_1 \in L$ implies that L_1 does not contain L' or L'' . Thus there exists $\mu_1 \in IR(L)$ st $\mu_1(L')=1$ $\mu \leq \mu_1$ (L) and also a $\mu_2 \in IR(L')$ st $\mu_2(L'')=1$ and $\mu \leq \mu_2$ (L'). But since $I(L)=IR(L)$ and this implies from above that $IR(L')=IR(L)$, $\mu_1(L'')=\mu_1(L'')=1$ or $\mu_1(L' \cap L'')=1=\mu_1(\emptyset)=0$, a contradiction. $L=L'$ and L is an algebra.

Note: It is well known even for abstract distributive lattices that $I(L)=IR(L)$ iff $L=L'$. (See Bourbaki [2], Huerta [6]).

Because of the importance of the last result we present an alternative approach which is of importance because of its relevance to lattice separation properties.

Theorem 3.2: Suppose L_1, L_2 are lattices of subsets of X st $L_2 \supseteq L_1$. If L_2 is disjunctive and L_1 is normal and if $\psi: IR(L_2) \rightarrow IR(L_1)$ where ψ is the restriction map, i.e. $\psi(v)=\mu$ the restriction of v to $A(L_1)$, then L_1 semi-separates L_2 .

Proof: Suppose $L_1 \in L_1$ and $L_2 \in L_2$ and $L_1 \cap L_2 = \emptyset$. Then $W_2(L_1) \cap W_2(L_2) = \emptyset$ and $\psi(W_2(L_2)) \cap W_1(L_1) = \emptyset$ for if $\mu = \psi(v)$ where $v \in W_2(L_2)$, then $v(L_2)=1$. Therefore $\mu(L_1)=v(L_1)=0$ and thus $\mu \notin W_1(L_1)$.

$\psi(W_2(L_2)) = \cap W_1(L_1; i \in I)$ an arbitrary index set I , and $L_1 \supseteq L_2$. This holds since $W_2(L_2)$ is closed and thus compact ($W_2(X) \supseteq W_2(L_2)$ and $W_2(X)$ is compact). Also ψ is continuous since $\psi^{-1}(W_1(L_1)) = W_2(L_1)$. L_1 is normal which is equivalent to $W_1(L_1)$ being T_2 by a known result (see Bourbaki [2]). Therefore since $W_2(L_2)$ is compact and ψ is continuous then $\psi(W_2(L_2))$ is compact and since $W_1(L_1)$ is T_2 , $\psi(W_2(L_2))$ is closed and thus $\psi(W_2(L_2)) = \cap W_1(L_1; i \in I)$ an arbitrary index set. Also since L_2 is disjunctive then since $\psi: IR(L_2) \rightarrow IR(L_1)$ is well defined L_1 is also disjunctive. But this implies that $L_1 \supseteq L_2$. Thus $\psi(W_2(L_2)) = \cap W_1(L_1; i \in I)$ $L_1 \supseteq L_2$.

Now look at $\psi(W_2(L_2)) \cap W_1(L_1) = (\cap W_1(L_1; i \in I)) \cap W_1(L_1) = \emptyset$, $i \in I$. Then by compactness $(\cap W_1(L_1; \alpha)) \cap W_1(L_1) = \emptyset$ $\alpha=1, 2, \dots, N$. Since L_1 is disjunctive, this implies that $\cap L_1 \alpha \supseteq L_2$ $\alpha=1, \dots, N$, $L_1' = \cap L_1 \alpha$ $\alpha=1, \dots, N$, $L_1' \in L_1$ and $L_1 \cap L_1' = \emptyset$. Thus L_1 semi-separates L_2 .

Corollary 3.1: If L is a lattice of subsets of X st $I(L)=IR(L)$ then L is an algebra.

Proof: Set $L_1=L$ and $L_2=A(L)$. Since $I(L)=IR(L)$, L is normal. Then the hypotheses of the theorem hold, thus L ss $A(L)$. Let $L' \cap L = \emptyset$ $L \in L$, then since $L' \in A(L)$ and L ss $A(L)$, this implies that $L' \in L$ $L=L'$, i.e. L is an algebra.

Note: Suppose L_2 is disjunctive and cc then $IR(\sigma, L_2)=IR(L_2)$ and if $v \in IR(L_2)$ then $\mu=\psi(v) \in I(\sigma, L_1)$, and if L_1 is a delta lattice and $S(L_1) \supseteq \sigma(L_1)$ then $\mu \in IR(\sigma, L_1)$, in which case if L_1 is also normal then L_1 ss L_2 by theorem 3.2.

Another application arises if L_2 is L_1 cb and L_1 is cc then $IR(\sigma, L_2) = IR(L_2)$. If L_2 is disjunctive and L_1 is a delta lattice, $\sigma(L_1) \supseteq S(L_1)$ and L_1 is normal, then theorem 3.2 can be applied and L_1 ss L_2 .

4. THE SPACES $I(\sigma^*, L)$ AND $IR(\sigma, L)$

In this section we wish to consider, analogous matters concerning the spaces $I(\sigma^*, L)$ and $IR(\sigma, L)$ to those considered earlier for $I(L)$ and $IR(L)$. First we obtain conditions when $I(\sigma^*, L) = IR(\sigma, L)$ implies L is an algebra. In this connection we introduce a definition.

Definition 4.1 The lattice of subsets of X is almost countably compact (acc) if $\mu \in IR(L')$ implies $\mu \in I(\sigma^*, L)$.

Remark: Clearly L cc implies L acc. It is easy to show that if L is normal and countably paracompact then L acc implies L cc: Namely let $\mu \in I(L)$ then $\mu \in I(L')$ and also there exists a $\mu_1 \in IR(L')$ st $\mu \leq \mu_1$ (L') or $\mu_1 \leq \mu$ (L). But L acc implies that $\mu_1 \in I(\sigma^*, L)$ and L normal and cp implies there exists $\mu_2 \in IR(\sigma, L)$ (see introduction) st $\mu_1 \leq \mu_2$ (L) and thus $\mu \leq \mu_2$ (L). Thus $\mu \in I(\sigma^*, L)$ and L is cc.

Theorem 4.1: If $I(\sigma^*, L) = IR(\sigma, L)$ and if L is acc then L is an algebra.

Proof: Let $\mu \in I(L)$ then there exists a $\mu_1 \in IR(L')$ st $\mu \leq \mu_1$ (L'). But L is acc, therefore $\mu_1 \in I(\sigma^*, L)$. But $I(\sigma^*, L) = IR(\sigma, L)$, thus $\mu_1 \in IR(\sigma, L)$ and since $\mu_1 \leq \mu$ (L) $\mu \in IR(\sigma, L)$. Thus $I(L) = IR(L)$, which implies that L is complemented or that is, L is an algebra.

Theorem 4.2: Consider the set $I(\sigma^*, L)$, then the lattice $V(\sigma, L)$ in $I(\sigma^*, L)$ is regular iff $IR(\sigma, L) = I(\sigma^*, L)$.

Proof: Assume that $I(\sigma^*, L) = IR(\sigma, L)$ and that $V(\sigma, L)$ is not regular. Then there exists $\mu \in I(\sigma^*, L)$ and $V(\sigma, L) \in V(\sigma, L)$ such that $\mu \notin V(\sigma, L)$ and $H = \{V(\sigma, L \sim) \mid V(\sigma, L \sim) \supseteq V(\sigma, L) \text{ or } \mu \in V(\sigma, L \sim)\}$ has the fip and thus there exists a $\mu_1 \wedge \in I(V(\sigma, L'))$ st $\mu_1 \wedge (V(\sigma, L \sim)) = 1$ $V(\sigma, L \sim) \in H$. In addition $\mu \wedge \in I(\sigma^*, V(\sigma, L))$ and $\mu \wedge \leq \mu_1 \wedge$ on $V(\sigma, L')$ or $\mu_1 \wedge \leq \mu \wedge$ on $V(\sigma, L)$. Therefore $\mu_1 \wedge \in I(\sigma^*, V(\sigma^*, L))$ and projecting downward $\mu_1 \leq \mu$ (L). But $I(\sigma^*, L) = IR(\sigma, L)$, therefore $\mu = \mu_1$ and $\mu \wedge = \mu_1 \wedge$ on $V(\sigma, L)$ projecting upward, also $\mu \wedge = \mu_1 \wedge \in IR(\sigma, V(\sigma, L))$. Since $\mu \wedge \in IR(\sigma, V(\sigma, L))$ and $\mu \wedge (V(\sigma, L')) = 1$ there exists a $V(\sigma, L_1 \sim)$ st $\mu \wedge (V(\sigma, L_1 \sim)) = 1$ and $V(\sigma, L') \supseteq V(\sigma, L_1 \sim)$ or $V(\sigma, L_1 \sim) \supseteq V(\sigma, L)$ and by definition of $\mu_1 \wedge$, $\mu \wedge (V(\sigma, L_1 \sim)) = \mu_1 \wedge (V(\sigma, L_1 \sim)) = 1$ a contradiction. $I(\sigma^*, L), V(\sigma, L)$ is regular.

Conversely let $I(\sigma^*, L), V(\sigma, L)$ be regular and let $\mu \in I(\sigma^*, L) \mu \notin V(\sigma, L)$, then $\mu \in V(\sigma, L')$ and $\mu(L') = 1$. Consider the projection upward, then $\mu \wedge \in I(\sigma^*, V(\sigma^*, L)) \mu \notin V(\sigma^*, L)$. Then since $I(\sigma^*, L), V(\sigma^*, L)$ is regular there exists $V(\sigma, L_1), V(\sigma, L_2) \in V(\sigma, L)$ st $V(\sigma, L_1) \supseteq V(\sigma, L) \mu \in V(\sigma, L_2)$ and $V(\sigma, L_1') \cap V(\sigma, L_2') = \emptyset$ or $V(\sigma, L_1) \cup V(\sigma, L_2) = I(\sigma^*, L)$. Then $\mu \wedge (V(\sigma, L_1)) = 1$ or $\mu \wedge (V(\sigma, L_2)) = 1$. Since $\mu \wedge (V(\sigma, L_2')) = 1$ $\mu \wedge (V(\sigma, L_2)) = 0$ and $\mu \wedge (V(\sigma, L_1)) = 1$ $V(\sigma, L_1') \supseteq V(\sigma, L)$ or $V(\sigma, L') \supseteq V(\sigma, L_1)$. This implies that $L' \supseteq L_1$ and $\mu(L_1) = 1$. Therefore $\mu \in IR(\sigma, L)$ and $I(\sigma^*, L) = IR(\sigma^*, L)$.

Theorem 4.3: Suppose $\tau L \supseteq E \supseteq A(L) \supseteq L$ then if τL is L cb or more generally if E (and thus $A(L)$) is L cb and a) $S(L) \supseteq \sigma(L)$ (in particular if $\rho(L) = \sigma(L)$) and L is delta or b) If L is complement generated (and not necessarily delta) then $IR(\sigma, L) = I(\sigma^*, L)$.

Proof: Let $\mu \in I(\sigma^*, L)$. Since $A(L)$ is L countably bounded $I(\sigma, L) = I(\sigma^*, L)$. Now let L be cg and $\mu(L') = 1, \mu \in I(\sigma, L) L \in L$, then $L' = \cup L_i$ $i=1, 2, \dots$ or $L = \cap L_i'$ $i=1, 2, \dots$. Then since $\mu \in I(\sigma, L)$, $0 = \lim \mu(L) = \lim \mu(\cap L_i') = 0$ where $i=1, 2, \dots, N$ for some N . Since $L' \supseteq \cup L_i$ $i=1, 2, \dots, N$ $\mu \in IR(\sigma, L)$ and $I(\sigma^*, L) = IR(\sigma, L)$.

Suppose instead that L is delta and $S(L) \supseteq \sigma(L)$ in particular $\rho(L) = \sigma(L)$. Since $A(L)$ is countably bounded by L $I(\sigma^*, L) = I(\sigma, L)$. Consider μ^* the outer measure induced by μ and its restriction to the μ^* -measurable sets. Then the μ^* -measurable sets include $\sigma(L)$ and thus $A(L)$, and μ^* is delta regular on such sets by the hypotheses $S(L) \supseteq \sigma(L)$ (or more particularly $\sigma(L) = \rho(L)$). Since L is delta this implies that $\mu \in IR(\sigma, L)$ and $I(\sigma^*, L) = IR(\sigma, L)$.

Note: If $I(\sigma^*, \mathbf{L}) = IR(\sigma, \mathbf{L})$ then $IR(\sigma, \mathbf{L}') = IR(\sigma, \mathbf{L})$ and $\mu_x \in IR(\sigma, \mathbf{L})$ and $\mu_x \in IR(\sigma, \mathbf{L}')$. Thus $\mathbf{L}$ and $\mathbf{L}'$ is disjunctive, thus $\tau \mathbf{L} \supseteq \mathbf{L}'$ and $\tau \mathbf{L}$ contains $\mathbf{L}$ and $\mathbf{L}'$, which implies that $\mathbf{L}$ is contained in the algebra of closed-open sets determined by the topology of $\mathbf{L}$ on X .

Definition 4.2: A lattice $\mathbf{L}$ is said to be prime complete if for any $\mu \in I(\sigma^*, \mathbf{L})$ $S(\mu) \neq \emptyset$.

Theorem 4.4: If $\mathbf{L}$ is disjunctive, and $W(\sigma, \mathbf{L})$ in $IR(\sigma, \mathbf{L})$ is prime complete, then for any $\mu \in I(\sigma^*, \mathbf{L})$, there exists a $\mu_1 \in IR(\sigma, \mathbf{L})$ st $\mu \leq \mu_1$ ($\mathbf{L}$).

Proof: Since $\mathbf{L}$ is disjunctive there exists a one to one correspondence between measures on $X, \mathbf{L}$ and $IR(\sigma, \mathbf{L}), W(\sigma, \mathbf{L})$. Thus let $\mu \in I(\sigma^*, \mathbf{L})$ then $\mu^\wedge \in I(\sigma^*, W(\sigma, \mathbf{L}))$ and since $W(\sigma, \mathbf{L})$ is prime complete $S(\mu^\wedge) \neq \emptyset$, then there exists a $\{\mu_1\} \in S(\mu^\wedge)$ $\mu_1 \in IR(\sigma, \mathbf{L})$. Further if $\mu(\mathbf{L}) = 1$ $\mathbf{L} \in \mathbf{L}$ then $\mu^\wedge(W(\sigma, \mathbf{L})) = \mu(\mathbf{L}) = 1$, and since $\mu_1 \in W(\sigma, \mathbf{L})$ $\mu_1(\mathbf{L}) = 1$ therefore $\mu \leq \mu_1$ ($\mathbf{L}$) with $\mu_1 \in IR(\sigma, \mathbf{L})$.

Theorem 4.5: If a) $\mathbf{L}$ is disjunctive and $W(\sigma, \mathbf{L})$ is prime complete or alternately b) $\mathbf{L}$ is normal and countably paracompact then if $I(\sigma^*, \mathbf{L}), V(\sigma, \mathbf{L})$ is T_1 then $I(\sigma^*, \mathbf{L}) = IR(\sigma, \mathbf{L})$.

Proof: Let $\mu \in I(\sigma^*, \mathbf{L})$ then by hypothesis a) or b) there exists a $\nu \in IR(\sigma, \mathbf{L})$ st $\mu \leq \nu$ ($\mathbf{L}$). Then since $I(\sigma^*, \mathbf{L}), V(\sigma, \mathbf{L})$ is T_1 there exists a $\mathbf{L}', \mathbf{L} \in \mathbf{L}$ st $\nu \in V(\sigma, \mathbf{L}')$ $\mu \notin V(\sigma, \mathbf{L}')$ which implies that $\mu(\mathbf{L}) = 1$ $\nu(\mathbf{L}) = 0$ a contradiction unless $\mu = \nu$ and $I(\sigma^*, \mathbf{L}) = IR(\sigma, \mathbf{L})$.

Theorem 4.6: If $I(\sigma^*, \mathbf{L}), V(\sigma, \mathbf{L})$ is T_1 then $\mu, \mu_1 \in I(\sigma^*, \mathbf{L})$ implies that if $\mu \neq \mu_1$ then neither $\mu \leq \mu_1$ ($\mathbf{L}$) or $\mu_1 \leq \mu$ ($\mathbf{L}$) holds. Conversely if neither $\mu \leq \mu_1$ ($\mathbf{L}$) or $\mu_1 \leq \mu$ ($\mathbf{L}$) $\mu \neq \mu_1$ holds then $I(\sigma^*, \mathbf{L}), V(\sigma, \mathbf{L})$ is T_1 .

Proof: Let $\mu, \mu_1 \in I(\sigma^*, \mathbf{L})$ $\mu \neq \mu_1$. Since $V(\sigma, \mathbf{L})$ is T_1 , this implies that there exists $V(\sigma, \mathbf{L}_1), V(\sigma, \mathbf{L}_2) \in V(\sigma, \mathbf{L})$ st $\mu \in V(\sigma, \mathbf{L}_1)$ $\mu_1 \notin V(\sigma, \mathbf{L}_1)$ $\mu \notin V(\sigma, \mathbf{L}_2)$ $\mu_1 \in V(\sigma, \mathbf{L}_2)$ or $\mu(\mathbf{L}_1) = 0$ $\mu_1(\mathbf{L}_1) = 1$ $\mu(\mathbf{L}_2) = 1$ $\mu_1(\mathbf{L}_2) = 0$. Thus neither $\mu \leq \mu_1$ ($\mathbf{L}$) or $\mu_1 \leq \mu$ ($\mathbf{L}$) can hold.

Conversely suppose $\mu, \mu_1 \in I(\sigma^*, \mathbf{L})$ $\mu \neq \mu_1$ and neither $\mu \leq \mu_1$ ($\mathbf{L}$) or $\mu_1 \leq \mu$ ($\mathbf{L}$) holds. This implies that there exists $\mathbf{L}_1, \mathbf{L}_2 \in \mathbf{L}$ st $\mu(\mathbf{L}_1) = 1$ $\mu_1(\mathbf{L}_1) = 0$ $\mu(\mathbf{L}_2) = 0$ $\mu_1(\mathbf{L}_2) = 1$ or $\mu(\mathbf{L}_1) = 0$ $\mu_1(\mathbf{L}_1) = 1$ $\mu(\mathbf{L}_2) = 1$ $\mu_1(\mathbf{L}_2) = 0$ or $\mu \in V(\sigma, \mathbf{L}_2)$ $\mu_1 \notin V(\sigma, \mathbf{L}_2)$ $\mu_1 \in V(\sigma, \mathbf{L}_1)$ $\mu \notin V(\sigma, \mathbf{L}_1)$ and $I(\sigma^*, \mathbf{L}), V(\sigma, \mathbf{L})$ is T_1 .

Definition 4.3: Denote by $\Pi(\sigma, \mathbf{L})$ the collection of premeasures that are sigma-smooth. A pre-measure $p \in \Pi(\mathbf{L})$ is defined on $\mathbf{L}$ and satisfies 1) $p(\emptyset) = 0$; 2) If $p(A) = 1$ $p(B) = 1$ $A, B \in \mathbf{L}$ then $p(A \cap B) = 1$; 3) If $p(B) = 1$ and $A \supseteq B$ where $A, B \in \mathbf{L}$ then $p(A) = 1$. It is sigma-smooth if $\{A_n\} \downarrow \emptyset$ $A_n \in \mathbf{L}$ then $\lim p(A_n) = 0$ as $n \rightarrow \infty$.

Definition 4.4: $\mathbf{L}$ is an I-lattice iff for every $p \in \Pi(\sigma, \mathbf{L})$ there exists a $\mu \in IR(\sigma, \mathbf{L})$ st $p \leq \mu$ ($\mathbf{L}$).

Theorem 4.7: Let $\mathbf{L}$ be an I-lattice which is also a delta lattice and suppose $I(\sigma^*, \mathbf{L}) = IR(\sigma, \mathbf{L})$, then $\mathbf{L}$ is complemented.

Proof: Assume $\mathbf{L}$ is not complemented, then for some $\mathbf{L} \in \mathbf{L}$ $\mathbf{L}' \notin \mathbf{L}$. Consider $H = \{\mathbf{L} \sim \mathbf{L}' \mid \mathbf{L} \sim \mathbf{L}', \mathbf{L}' \in \mathbf{L}\}$. Then since $\mathbf{L}$ is delta, H has the countable intersection property since $\mathbf{L}' \neq \emptyset$. Thus there exists a $p \in \Pi(\sigma, \mathbf{L})$ associated with H . Since $\mathbf{L}$ is I-lattice there exists $\mu \in IR(\sigma, \mathbf{L})$ st $p \leq \mu$ ($\mathbf{L}$). Also $I(\sigma^*, \mathbf{L}) = IR(\sigma, \mathbf{L})$ implies that $IR(\sigma, \mathbf{L}) = IR(\sigma, \mathbf{L}')$.

Now if $\mathbf{L} \sim \mathbf{L}'$ then $\mathbf{L} \sim \mathbf{L}' \cap \mathbf{L} \neq \emptyset$, since $\mathbf{L}' \notin \mathbf{L}$. Thus $\mu(\mathbf{L}) = 1$ since associated with μ is an ultrafilter. But since $\mu \in IR(\sigma, \mathbf{L}')$ $\mu(\mathbf{L}) = 0$, a contradiction and hence $\mathbf{L}$ is complemented.

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