RESEARCH NOTES

A REMARK ON RHOADES FIXED POINT THEOREM FOR NON-SELF MAPPINGS

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ABSTRACT. Let X be a Banach space, K a non-empty closed subset of X and $T:K \rightarrow X$ a mapping satisfying the contractive definition (1.1) below and the condition $T(\partial K) \subseteq K$. Then T has a unique fixed point in K . This result improves Theorem of Rhoades [1] and generalizes the corresponding theorem of Assad [2].

KEY WORDS AND PHRASES. Fixed point, Cauchy sequence, complete metric space.

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1. INTRODUCTION.

Let X be a Banach space and K a closed subset of X . In many applications the domain of a considered function is K , but the codomen is not entirely included in K . So it is of interest to amplify a class of such mappings which have a fixed point. Rhoades [1] introduced a class of non-self mappings T of K into X which satisfy the following contractive definition

$$d(Tx, Ty) \leq h \cdot \max\{d(x, y)/2, d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/q\}, \quad (1.1)$$

where h and q are reals satisfying $0 < h < 1, q \geq 1 + 2h$. Rhoades proved that if $T(\partial K) \subseteq K$, then T has a unique fixed point. As pointed out by Rhoades [1, p. 459], the method of proof used in his Theorem 1 does not extend to more general contractive definitions.

The purpose of this note is to extend the result of Rhoades [1] to a class of non-self mappings of K which satisfy the following contractive definition:

There exists a constant $h, 0 < h < 1$, such that for each $x, y \in K$,

$$d(Tx, Ty) \leq h \cdot \max\{d(x, y)/a, d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/(a + h)\}, \quad (1.2)$$

where a is a real number satisfying $a \geq 1 + (2h^2)/(1 + h)$.

Note that if T satisfies the condition (1.1) then T satisfies condition (1.2) with $a = 1 + h$.

Using a new method of proof we proved the result which is an improvement of the Theorem of Rhoades [1] and generalization of the Theorem of Assad [2].

2. MAIN RESULT.

In this paper we shall use the fact that, if $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$, the boundary of K , such that $d(x, z) + d(z, y) = d(x, y)$.

THEOREM 2.1. Let X be a Banach space, K a non-empty closed subset of X and $T:K \rightarrow X$ a

mapping satisfying (1.2) on K and such that $T(\partial K) \subseteq K$. Then T has a unique fixed point in K at which T is continuous.

PROOF. Let $x_0 \in K$ be arbitrary point. Define two sequences $\{x_n\}$ and $\{x'_n\}$ satisfying the following conditions:

- (i) $x'_{n+1} = Tx_n$,
- (ii) $x_n = x'_n$ if $x'_n \in K$,
- (iii) $x_n \in \partial K$ and $d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x'_{n-1}, x'_n)$ if $x'_n \notin K$.

Let $P = \{x_k \in \{x_n\}; x_k = x'_k\}$ and $Q = \{x_k \in \{x_n\}; x_k \neq x'_k\}$. Note that $\{x_n\} \subseteq K$ and if $x_n \in Q$, then x_{n-1} and x_{n+1} belong to P , since by $T(\partial K) \subseteq K$ we cannot have two consecutive points of $\{x_n\}$ in Q .

From (1.2) it is easy to obtain $d(x'_n, x'_{n+1}) \leq h \cdot d(x_{n-1}, x_n)$.

We shall estimate $d(x_n, x_{n+1})$. Actually, we have three cases to consider:

Case I: $x_n, x_{n+1} \in P$,

Case II: $x_n \in P, x_{n+1} \in Q$,

Case III: $x_n \in Q, x_{n+1} \in P$.

It is easily seen that Cases I and II imply

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n). \quad (2.1)$$

Case III. $x_n \in Q, x_{n+1} \in P$. We shall show that

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-2}, x_{n-1}). \quad (2.2)$$

If $d(x_n, x_{n+1}) \leq d(x_{n-1}, x'_n)$, then (2.2) holds by Case II, since $x_n \in Q$ implies $x_{n-1} \in P$. Assume now that

$$d(x_n, x_{n+1}) > d(x_{n-1}, x'_n). \quad (2.3)$$

Since $x_n \in Q$ and is a convex linear combination of x_{n-1} and x'_n it follows

$$d(x_n, x_{n+1}) \leq \max\{d(x'_n, x_{n+1}), d(x_{n-1}, x_{n+1})\}. \quad (2.4)$$

Using (1.1) and (2.3) we obtain

$$d(x'_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq h \cdot \max\{d(x_{n-1}, x_n)/a, d(x_n, x_{n+1}), 2d(x_n, x_{n+1})/(a+h)\}$$

and hence, as $a \geq 1$,

$$d(x'_n, x_{n+1}) \leq 2h \cdot d(x_n, x_{n+1})/(1+h). \quad (2.5)$$

Then from (2.4) and (2.5) we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_{n+1}). \quad (2.6)$$

Using again (1.1), (2.3) and the triangle inequality we have

$$\begin{aligned} d(x_{n-1}, x_{n+1}) &= d(Tx_{n-2}, Tx_n) \leq h \max\{[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)]/a, \\ &\quad d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), [d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)]/(a+h)\}. \end{aligned} \quad (2.7)$$

Suppose that (2.7) implies

$$d(x_{n-1}, x_{n+1}) \leq h \cdot [d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)]/(a+h).$$

Then we have

$$d(x_{n-1}, x_{n+1}) \leq h \cdot [d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)]/a.$$

Since by (2.6) $d(x_{n-1}, x_{n+1}) \leq h \cdot d(x_{n-2}, x_{n-1})$ immediately implies (2.2), and $d(x_{n-1}, x_{n+1}) \leq h \cdot d(x_n, x_{n+1})$ is in contradiction with (2.6), we may suppose that (2.7) implies

$$d(x_{n-1}, x_{n+1}) \leq h \cdot [d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)]/a. \quad (2.8)$$

Assume now that $d(x_{n-1}, x_{n+1}) > [2h/(1+h)] \cdot d(x_{n-1}, x'_n)$. Then by (2.1) we get

$$d(x_n, x'_n) \leq [1 - 2h/(1+h)] \cdot d(x_{n-1}, x'_n) \leq [(1-h)/(1+h)] \cdot h \cdot d(x_{n-2}, x_{n-1})$$

and so by the triangle inequality and (2.5) we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &\leq [h(1-h)/(1+h)] \cdot d(x_{n-2}, x_{n-1}) + [2h/(1+h)] \cdot d(x_n, x_{n+1}). \end{aligned}$$

Hence it follows that $d(x_n, x_{n+1}) \leq h \cdot d(x_{n-2}, x_{n-1})$, i.e., the relation (2.2).

Assume now that $d(x_{n-1}, x_n) \leq [2h/(1+h)] \cdot d(x_{n-1}, x'_n)$. Then from (2.1) and (2.8) we get

$$d(x_{n-1}, x_{n+1}) \leq h \cdot [1 + 2h^2/(1+h)] \cdot d(x_{n-2}, x_{n-1})/a \leq h \cdot d(x_{n-2}, x_{n-1}),$$

since by hypothesis $a \geq 1 + 2h^2/(1+h)$. So we proved (2.2).

By (2.1) and (2.2) we conclude that in all cases

$$d(x_n, x_{n+1}) \leq h \cdot \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}.$$

Now it is easily shown by induction that, for $n \geq 2$,

$$d(x_n, x_{n+1}) \leq h^{(n/2)} \max\{d(x_0, x_1), d(x_1, x_2)\},$$

where $(n/2)$ is the greatest integer not exceeding $n/2$. Hence for $m > n > N$,

$$d(x_n, x_m) \leq \sum_{i=N}^{\infty} d(x_i, x_{i+1}) \leq [2h^{(N/2)}/(1-h)] \cdot \max\{d(x_0, x_1), d(x_1, x_2)\},$$

so that $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subseteq K$ and K is closed, $\{x_n\}$ converges to some point $p \in X$.

Let $\{x_{n(k)+1}\} \subseteq P$ be an infinite subsequence of $\{x_n\}$. From (1.2)

$$\begin{aligned} d(p, Tp) &\leq d(p, Tx_{n(k)}) + d(Tx_{n(k)}, Tp) \leq (p, Tx_{n(k)}) + h \cdot \max\{d(x_{n(k)}, p)/a, \\ &\quad d(x_{n(k)}, x_{n(k+1)}), d(p, Tp), [d(x_{n(k)}, Tp) + d(p, x_{n(k)+1})]/(a+h)\}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ yields $d(p, Tp) \leq h \cdot d(p, Tp)$. Hence $Tp = p$. Condition (1.2) implies uniqueness.

Mappings which satisfy (1.2) may be discontinuous but at a fixed point they are continuous. For if $y_n \rightarrow p = Tp$, then from (1.2) we have

$$\begin{aligned} d(Ty_n, p) &= d(Ty_n, Tp) \\ &\leq h \cdot \max\{d(y_n, p)/a, [d(y_n, p) + d(p, Ty_n), [d(y_n, p) + d(p, Ty_n)]/(a+h)\}. \end{aligned}$$

Hence $\limsup d(Ty_n, p) \leq h \cdot \limsup d(p, Ty_n)$. Hence $Ty_n \rightarrow p$. This completes the proof.

The following result readily follows from Theorem 2.1.

COROLLARY. Let X be a Banach space, K a non-empty closed subset of X and $T:K \rightarrow X$ a mapping satisfying

$$d(T^k x, T^k y) \leq h \cdot \max\{d(x, y)/a, d(x, T^k x), d(y, T^k y), [d(x, T^k y) + d(y, T^k x)]/(a + h)\} \quad (2.9)$$

for all $x, y \in K$, where k is a positive integer and a, h constants such that $0 < h < 1$ and $a > 1 + 2h^2/(1 + h)$. If $T(\partial K) \subset K$, then T has a unique fixed point in K .

Theorem 2.1 can easily be extended to multi-valued mappings. Let (X, d) be a metric space
 (11) $BN(X)$ the set of all bounded subsets of X . For $A, B \in BN(X)$ set $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$.

Now we can state our result.

THEOREM 2.2. Let X be a Banach space, K a non-empty closed subset of X and $F:K \rightarrow BN(X)$ a multi-valued mapping satisfying

$$\delta(Fx, Fy) \leq h \cdot \max\{d(x, y)/a, \delta(x, Fx), \delta(y, Fy), [D(x, Fy) + D(y, Fx)]/(a + h)\}$$

for all $x, y \in K$, where a and h are reals satisfying $0 < h < 1$, $a \geq 1 + 2h^2/(1 + h)$. If $Fx \subseteq K$ for all $x \in \partial K$, then F has a unique stationary point in K (i.e., there is some $\xi \in K$ such that $F\xi = \{\xi\}$).

The proof of Theorem 2.2 is omitted, since it follows the same arguments as those of Theorem 3 of [2] and Theorem 2.1 above.

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