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FEEDBACK REGULATION OF LOGISTIC GROWTH

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Abstract

Sufficient conditions are obtained for the global asymptotic stability of the positive equilibrium of a regulated logistic growth with a delay in the state feedback of the control modelled by

$$\frac{dn(t)}{dt} = rn(t)\left[1 - \left(\frac{a_1 n(t) + a_2 n(t-\tau)}{K}\right) - cu(t)\right]$$
$$\frac{du(t)}{dt} = -au(t) + bn(t-\tau)$$

where u denotes an indirect control variable, $r, a_2, \tau, a, b, c \in (0, \infty)$ and $a_1 \in [0, \infty)$.

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1. Introduction

The autonomous ordinary differential equation

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \frac{N(t)}{K}\right], \qquad r, K \in (0, \infty) \tag{1.1}$$

has been a basis for the development of several models of dynamics of populations in mathematical ecology. It is an elementary fact that if $N(0) > 0$ then the corresponding solution of (1.1) satisfies

$$\lim_{t\to\infty} N(t) = K \tag{1.2}$$

and the convergence in (1.2) is monotonic. It has been found that such a monotonic convergence to the equilibrium is not realistic in certain laboratory populations in temporally uniform environments (Hutchinson [9]); also some of the negative feedback effects (such as the accumulation of toxic residuals etc.) act with some time delay (Volterra [13]). One of the possible ways of modifying (1.1) so as to model these additional features is to consider a time lagged equation of the form

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \left(\frac{a_1 N(t) + a_2 N(t-\tau)}{K}\right)\right] \tag{1.3}$$

where $a_1, a_2, K, \tau \in (0, \infty)$. Equations of the form (1.3) have been discussed by several authors in the literature on population dynamics.

Along with (1.3) one considers an initial condition of the form

$$N(s) = \phi(s) \geq 0; \qquad \phi(0) > 0; \qquad \phi \in C([-\tau, 0], R_+). \tag{1.4}$$

Solutions of (1.3) and (1.4) satisfy

$$N(t) = N(0) \exp\left[r \int_0^t \left\{1 - \left(\frac{a_1 N(s) + a_2 N(s-\tau)}{K}\right)\right\} ds\right], \quad t > 0$$

from which one can see that solutions of (1.3) and (1.4) are defined for all $t \geq 0$ and also satisfy $N(t) > 0$ for all $t \geq 0$. It is known that if $a_1 > a_2$ then all solutions of (1.3)-(1.4) satisfy

$$\lim_{t \to \infty} N(t) = N^* = \frac{K}{a_1 + a_2} \tag{1.5}$$

and the convergence in (1.5) is unconditional on the size of delay (for details see Lenhart and Travis [12] or Gopalsamy [4]). We shall suppose that we have a situation where the equilibrium N^* of (1.5) is not the desirable one (or affordable) and a smaller value of N^* is required; thus we are required to alter the system (1.3) structurally so as to make the population stabilise at value lower than that in (1.5). One of the methods of accomplishing this is to introduce a *"feedback"* control variable and this can be implemented by means a biological control or some harvesting procedure. We formulate one such model below. Stability of feedback control systems has been discussed in the books by LaSalle and Lefschetz [10], Lefschetz [11] and Aizerman and Gantmacher [1].

2. Feedback regulation

We consider now the initial value problem

$$\left.\begin{aligned} \frac{dn(t)}{dt} &= rn(t)\left[1 - \left(\frac{a_1 n(t) + a_2 n(t-\tau)}{K}\right) - cu(t)\right] \\ \frac{du(t)}{dt} &= -au(t) + bn(t); \qquad a, b, c \in (0, \infty) \\ n(s) &= \phi(s) \geq 0; \quad \phi(0) > 0; \quad \phi \in C([-\tau, 0], R_+) \\ u(0) &= u_o > 0 \end{aligned}\right\} \tag{2.1}$$

where u denotes an *"indirect control"* variable (Aizerman and Gantmacher [1] or Lefschetz [11]). It is not difficult to see that solutions of (2.1) are defined for all $t \geq 0$ and also satisfy

$$n(t) > 0, \qquad u(t) > 0 \qquad \text{for} \qquad t > 0.$$

The controlled system (2.1) has a positive equilibrium (n^*, u^*) defined by

$$\left(\frac{a_1 + a_2}{K} + \frac{bc}{a}\right) n^* = 1; \qquad u^* = \frac{b}{a} n^*.$$

The following preparation will be useful in the proof of our result on the global attractivity of (n^*, u^*).

First we verify that all solutions of (2.1) remain bounded for all $t \geq 0$. Suppose for instance that

$$\limsup_{t\to\infty} n(t) = \infty;$$

let $\{t_m\}$ be a sequence such that

$$t_m \to \infty, \qquad n(t_m) \to \infty \quad \text{as} \quad m \to \infty \quad \text{and} \quad \left.\frac{dn(t)}{dt}\right|_{t_m} \geq 0; \tag{2.2}$$

it will follow from the first of (2.1) that

$$\left.\frac{dn}{dt}\right|_{t_m} < rn(t_m)\left[1 - \frac{a_1 n(t_m)}{K}\right] < 0 \tag{2.3}$$

if m is large enough and so (2.3) will contradict (2.2). Thus we conclude

$$\limsup_{t\to\infty} n(t) < \infty.$$

One can in fact obtain explicit bounds for n and we do not need these for our work below. The second of (2.1) can be written as

$$\frac{d}{dt}[u(t)e^{at}] = bn(t)e^{at}$$

leading to

$$\begin{aligned} u(t)e^{at} &= u(0) + b\int_0^t n(s)e^{as}ds \\ &\leq u(0) + \tilde{n}\frac{b}{a}(e^{at} - 1); \qquad \tilde{n} = \sup_{t\geq 0} n(t) \end{aligned} \tag{2.4}$$

and hence

$$u(t) \leq u(0)e^{-at} + \frac{b}{a}\tilde{n}(1 - e^{-at}); \qquad t > 0 \tag{2.5}$$

from which the uniform boundedness of u follows. For convenience in the following we define new variables x, y, σ as follows:

$$\left.\begin{aligned} x(t) &= u(t) - u^* \\ y(t) &= -\frac{n^*}{r}\left[ln\frac{n(t)}{n^*} - \frac{cr}{a}x(t)\right] \\ \sigma(t) &= \frac{cr}{a}x(t) - \frac{r}{n^*}y(t). \end{aligned}\right\} \tag{2.6}$$

One can verify by direct calculation that

$$\frac{dx(t)}{dt} = -ax(t) + b\phi(\sigma(t)) \tag{2.7}$$

$$\frac{d\sigma(t)}{dt} = -crx(t) - \frac{a_1 r}{K}\phi(\sigma(t)) - \frac{a_2 r}{K}\phi(\sigma(t-\tau)) \tag{2.8}$$

$$\frac{dy(t)}{dt} = \left(\frac{a_1}{K} + \frac{bc}{a}\right)n^*\phi(\sigma(t)) + \frac{a_2}{K}n^*\phi(\sigma(t-\tau)) \tag{2.9}$$

where

$$\phi(\sigma(t)) = n^*\left[e^{\sigma(t)} - 1\right]. \tag{2.10}$$

Theorem 2.1. *Suppose r, K, a_1, a_2, b, c are positive numbers such that $a_1 > a_2 \geq 0$ and $\tau \geq 0$. Then all positive solutions of (2.1) satisfy*

$$\lim_{t\to\infty} n(t) = n^*; \qquad \lim_{t\to\infty} u(t) = u^*. \tag{2.11}$$

Proof. It is sufficient to show that

$$\lim_{t\to\infty} x(t) = 0 \qquad \textit{and} \qquad \lim_{t\to\infty} y(t) = 0 \tag{2.12}$$

due to our definition in (2.6). We shall prove (2.12). We consider a Lyapunov functional $V = V(x(\cdot), \sigma(\cdot))$ defined by

$$V(x,\sigma)(\cdot) = Bx^2(t) + \beta \int_o^{\sigma(t)} \phi(s)ds + \delta \int_{t-\tau}^{t} \phi^2(\sigma(s))ds \tag{2.13}$$

where B, β and δ are positive numbers to be selected below suitably. Note that by the definition ϕ in (2.10), V is nonnegative. Calculating the rate of change of V along the solutions of (2.7)-(2.10),

$$\begin{aligned}
\frac{dV}{dt} &= -2Bax^2(t) + x(t)\phi(\sigma(t))[2Bb - \beta cr] \\
&\quad + \phi^2(\sigma(t))\left[\delta - \frac{\beta a_1 r}{K}\right] \\
&\quad + \phi^2(\sigma(t-\tau))[-\delta] \\
&\quad + \phi(\sigma(t))\phi(\sigma(t-\tau))\left[-\frac{\beta a_2 r}{K}\right] \\
&\leq -2Bax^2(t) + x(t)\phi(\sigma(t))[2Bb - \beta cr] \\
&\quad + \phi^2(\sigma(t))\left[\delta - \frac{\beta a_1 r}{K}\right] + \phi^2(\sigma(t-\tau))[-\delta] \\
&\quad + \frac{\beta a_2 r}{K}\frac{1}{2}\phi^2(\sigma(t)) + \frac{\beta a_2 r}{K}\frac{1}{2}\phi^2(\sigma(t-\tau)).
\end{aligned} \tag{2.14}$$

Let us now choose δ such that

$$\delta = \frac{1}{2}\frac{\beta a_2 r}{K} \tag{2.15}$$

and define η as follows:

$$\eta^2 = \frac{\beta r}{K}(a_1 - a_2); \tag{2.16}$$

note that this definition of η is possible since by assumption $a_1 > a_2$. One can now simplify (2.14) to the form

$$\frac{dV}{dt} \leq -\left[Bax^2(t) + Bax^2(t) - 2x(t)\phi(\sigma(t))\left(Bb - \frac{\beta cr}{2}\right) + \eta^2\phi^2(\sigma(t))\right] \tag{2.17}$$

We now choose B such that

$$Bb - \frac{\beta cr}{2} = \sqrt{Ba}\eta \tag{2.18}$$

where β is any positive number; we observe that it is sufficient to choose B to be the positive root of the quadratic equation

$$B^2b^2 - B[a\eta^2 + b\beta cr] + \frac{\beta^2c^2r^2}{4} = 0. \tag{2.19}$$

For this choice of B, we have from (2.17),

$$\frac{dV}{dt} \leq -\left[Bax^2(t) + \left\{\sqrt{Ba}x(t) - \eta\phi(\sigma(t))\right\}^2\right] \tag{2.20}$$

leading to

$$V(x,\sigma)(t) + \int_0^t \left[Bax^2(s) + \left\{\sqrt{Ba}x(s) - \eta\phi(\sigma(s))\right\}^2\right]ds \leq V(x,\sigma)(0). \tag{2.21}$$

It follows from (2.21) that $x^2 \in L_1(0,\infty)$. We want to verify that x is uniformly continuous on $(0,\infty)$ and for this it is sufficient that $\dot{x}$ is uniformly bounded on $(0,\infty)$. From (2.7) we can see that $\dot{x}$ will be uniformly bounded if both x is uniformly bounded and σ is uniformly bounded above. The uniform boundedness of x is immediate from (2.21) since (2.21) implies

$$Bx^2(t) \leq V(x,\sigma)(t) \leq V(x,\sigma)(0).$$

Suppose

$$\limsup_{t\to\infty} \sigma(t) = \infty;$$

then there exists a sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$\left.\frac{d\sigma}{dt}\right|_{t_m} \geq 0; \qquad \sigma(t_m - \tau) \leq \sigma(t_m), \qquad \sigma(t_m) \to \infty \quad \text{as} \quad m \to \infty. \tag{2.22}$$

We derive from (2.8) and (2.22) that

$$\left.\frac{d\sigma}{dt}\right|_{t_m} \leq -crx(t_m) - \frac{r}{K}(a_1 - a_2)\phi(\sigma(t_m)) \tag{2.23}$$

and (2.23) contradicts (2.22) for large enough m. Thus we conclude that

$$\limsup_{t\to\infty} \sigma(t) < \infty. \tag{2.24}$$

From (2.24) and the uniform boundedness of x we can conclude that $\dot{x}$ is uniformly bounded with the implication that x is uniformly continuous on $(0,\infty)$. It is easy to infer from (2.21) that $x^2 \in L_1(0,\infty)$; by Barbalat's lemma (see Corduneanu [3]) we conclude that

$$\lim_{t\to\infty} x(t) = 0. \tag{2.25}$$

The uniform continuity of $\sqrt{Ba}x(t) - \eta\phi(\sigma(t))$ can be verified from the above analysis of x and σ. Again from (2.21) we have

$$\left[\sqrt{Ba}x(t) - \eta\phi(\sigma(t))\right]^2 \in L_1(0,\infty)$$

and threrfore by Barbalat's lemma (Corduneanu [3]) we can conclude

$$\lim_{t\to\infty}\left[\sqrt{Ba}x(t) - \eta\phi(\sigma(t))\right] = 0. \tag{2.26}$$

Since $x(t) \to 0$ as $t \to \infty$, it will then follow from (2.26) that

$$\lim_{t\to\infty}\sigma(t) = 0 \qquad \textit{and} \qquad \lim_{t\to\infty} y(t) = 0. \tag{2.28}$$

This completes the proof.

3. Stabilisation and bifurcation control

It is known that the positive equilibrium of the delay logistic equation

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right] \tag{3.1}$$

becomes linearly unstable when $r\tau \geq \frac{\pi}{2}$ and for small positive $(r\tau - \frac{\pi}{2})$ there exists a periodic solution solution of (3.1) bifurcating from the positive equilibrium (Hassard et al. [8]). We suppose that it is required to stabilise the system (3.1) possibly at a different equilibrium and thereby avoid the bifurcation to periodicity. Accordingly we consider the feedback control system

$$\left.\begin{aligned} \frac{dn(t)}{dt} &= rn(t)\left[1 - \frac{n(t-\tau)}{K} - cu(t)\right] \\ \frac{du(t)}{dt} &= -au(t) + bn(t) \end{aligned}\right\} \tag{3.2}$$

where $r, K, c, a, b \in (0,\infty), \tau \in [0,\infty)$ and (3.2) is supplemented with initial conditions of the type (2.1). The system (3.2) has a positive equilibrium (n^*, u^*) where

$$u^* = \frac{b}{a}n^*, \qquad \left(\frac{1}{K} + \frac{bc}{a}\right)n^* = 1. \tag{3.3}$$

In terms of the variables x, y, σ defined in (2.6) we rewrite the system (3.2) as follows:

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= -ax(t) + b\phi(\sigma(t)); \qquad \phi(\sigma(t)) = n^*[e^{\sigma(t)} - 1] \\ \frac{d\sigma(t)}{dt} &= -crx(t) - \frac{r}{K}\phi(\sigma(t-\tau)). \end{aligned}\right\} \tag{3.4}$$

Theorem 3.1. *Assume that the parameters of the system (3.2) satisfy*

$$\frac{bc}{a} > \frac{1}{K} \qquad \textit{and} \qquad a > (1+\sqrt{2})r. \tag{3.5}$$

Then the positive equilibrium (n^*, u^*) of (3.2) is linearly asymptotically stable irrespective of the size of the delay τ.

Proof. The equilibrium (n^*, u^*) of (3.2) is linearly asymptotically stable if the trivial solution of the system (3.4) is linearly asymptotically stable. Such a linear system is

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= -ax(t) + bn^*S(t) \\ \frac{dS(t)}{dt} &= -crx(t) - \frac{rn^*}{K}S(t-\tau) \end{aligned}\right\} \tag{3.6}$$

whose characteristic equation is

$$\begin{vmatrix} \lambda + a & -bn^* \\ cr & \lambda + \frac{r}{K}n^* e^{-\lambda\tau} \end{vmatrix} = 0$$

or equivalently

$$\lambda^2 + \lambda\left(a + \frac{r}{K}n^* e^{-\lambda\tau}\right) + \frac{arn^*}{K}e^{-\lambda\tau} + bcrn^* = 0. \tag{3.7}$$

If $\tau = 0$, (3.7) becomes

$$\lambda^2 + \lambda\left(a + \frac{r}{K}n^*\right) + n^*\left(bcr + \frac{ar}{K}\right) = 0$$

whose roots have negative real parts. Thus the trivial solution of the linear system (3.6) is asymptotically stable when $\tau = 0$. By the continuity of the roots of (3.7) on the parameters (for details see Cooke and Grossman [2]), it will follow that all the roots of (3.7) will have negative real parts for small positive values of τ. We want to find sufficient conditions so that the real parts of the roots of (3.7) will have negative real parts whatever the size of the delay τ.

We note first $\lambda = 0$ is not a root of (3.7). If the root of (3.7) have zero or positive real parts, then for some positive $\tau, \lambda = \pm iw, w > 0$ are roots of (3.7). If we can show that $\lambda = \pm iw$ cannot be roots of (3.7), then it will follow that all roots of (3.7) lie on the left half of the complex plane for all $\tau > 0$. For instance if $\lambda = +iw$ is a root of (3.7) then

$$-w^2 + iw\left\{a + \frac{r}{K}n^* e^{-iw\tau}\right\} + \frac{ar}{K}n^* e^{-iw\tau} + bcrn^* = 0. \tag{3.8}$$

Separating the real and imaginary parts of (3.8),

$$\left.\begin{aligned} w^2 - bcrn^* &= w\frac{r}{K}n^* \sin w\tau + \frac{ar}{K}n^* \cos w\tau \\ -aw &= w\frac{r}{K}n^* \cos w\tau - \frac{ar}{K}n^* \sin w\tau. \end{aligned}\right\} \tag{3.9}$$

Squaring and adding the respective sides of (3.9),

$$w^4 + w^2\left[a^2 - 2bcrn^* - \left(\frac{rn^*}{K}\right)^2\right] + (bcrn^*)^2 - \left(\frac{ar}{K}n^*\right)^2 = 0. \tag{3.10}$$

It can be found that if

$$(bcrn^*)^2 > \left(\frac{ar}{K}n^*\right)^2 \tag{3.11}$$

$$a^2 > 2bcrn^* + \left(\frac{rn^*}{K}\right)^2 \tag{3.12}$$

then (3.10) will have no positive w^2 and hence (3.7) cannot have a pure imaginary root. By the first of (3.5), (3.11) is satisfied. We note from the second of (3.5) and the fact

$$bcn^* < a \qquad and \qquad n^* < K \tag{3.13}$$

that

$$\begin{aligned} a^2 - 2bcrn^* - \left(\frac{rn^*}{K}\right)^2 &> a^2 - 2ra - r^2 \\ &= (a-r)^2 - 2r^2 > 0 \end{aligned} \tag{3.14}$$

and hence (3.12) holds. Thus (3.10) has no real roots; it will then follow that (3.7) cannot have pure imaginary roots for any positive τ. We can conclude that a delay induced switching from stability to instablity cannot take place. Thus the absolute (or delay independent) linear stability of the positive equilibrium of (3.2) follows. This completes the proof.

We remark that without the indirect control u in (3.2), the system (3.1) can have periodic solutions arising through a Hopf-type bifurcation for suitable delays. We have illustrated that an appropriate indirect feedback control can be used to avoid the occurance of a Hopf-type bifurcation and stabilise a logistic growth by structurally altering the logistic growth.

4. Feedback control with time delay

In this section we continue with a discussion of the dynamics of the system (3.1) together with a feedback control when there is a delay in the state feedback of the control variable. We shall consider the following system

$$\left.\begin{aligned} \frac{dn(t)}{dt} &= rn(t)\Big[1 - \frac{n(t-\tau)}{K} - cu(t)\Big] \\ \frac{du(t)}{dt} &= -au(t) + bn(t-\tau) \end{aligned}\right\} \qquad 4.1$$

in which the feedback to the stabilising control u involves a time lag τ. We assume r, a, b, K, τ are positive numbers. One can see that (4.1) has a nontrivial steady state (n^*, u^*) where

$$n^* = \frac{aK}{a + Kbc}; \quad u^* = \frac{bK}{a + Kbc}. \qquad 4.2$$

We shall assume that the positive numbers r, K, a, b, c, τ are such that

$$\frac{bcK}{a} e^{r\tau} < 1. \qquad 4.3$$

A consequence of (4.3) is that there exists $\epsilon > 0$ satisfying

$$c\Big(\frac{bK}{a} e^{r\tau} + \epsilon\Big) < 1. \qquad 4.4$$

For convenience in the following we define α as follows:

$$\alpha = 1 - c(\frac{bK}{a} e^{r\tau} + \epsilon). \qquad 4.5$$

We shall assume that suitable initial conditions of

$$\left.\begin{aligned} &n(s) = \phi(s) \geq 0, \quad u(0) = u_o > 0, \\ &\phi(0) > 0, \quad \phi \in C\big[[-\tau, 0], R_+\big], \end{aligned}\right\} \qquad 4.6$$

are provided for (4.1). One can prove by the method of steps that solutions of (4.1) and (4.6) are defined for all $t \geq 0$ and remain positive for $t \geq 0$.

The following Lemma provides a priori estimates of the solutions of (4.1) and (4.6).

Lemma 4.1 . *Let ϵ be a positive number such that (4.4) holds and let $(n(t), u(t))$ denote arbitrary positive solution of (4.1) and (4.6). Then there exists a $T^* > 0$ such that*

$$\left.\begin{aligned} \ell \leq & n(t) \leq L \\ m \leq & u(t) \leq M \end{aligned}\right\} \quad \text{for} \quad t \geq T^*; \tag{4.7}$$

where

$$\left.\begin{aligned} \ell = & \alpha K exp\big[(\alpha - e^{r\tau})r\tau\big]; \quad L = Ke^{r\tau} \\ m = & \frac{b\ell}{a} - \epsilon \qquad M = \frac{bL}{a} + \epsilon \end{aligned}\right\} \tag{4.8}$$

(If necessary choose ϵ small enough so that $m > 0$).

Proof. By the positivity of u, we have from (4.1) that

$$\frac{dn(t)}{dt} \leq rn(t)\Big\{1 - \frac{n(t-\tau)}{K}\Big\} \quad \text{for} \quad t > 0. \tag{4.9}$$

There are two possibilities; n is oscillatory about K or n is nonoscillatory about K. If $n(t)$ is not oscillatory about K then there exists a T_o such that

$$\text{either} \quad n(t) > K \quad \text{for} \quad t \geq T_o \quad \text{or} \quad n(t) < K \quad \text{for} \quad t \geq T_o. \tag{4.10}$$

If the second alterative holds then we have

$$n(t) \leq Ke^{r\tau} \quad \text{for} \quad t \geq T_o.$$

Suppose $n(t) > K$ for $t > T_o$; then

$$\frac{dn(t)}{dt} < 0, \qquad \text{for} \quad t > T_o + \tau$$

and hence for some $n_o > 0$, we have

$$n(t) \to n_o, \qquad \text{as} \quad t \to \infty.$$

One can show that $n_o \leq K$ (the details of proof omitted) from which it will follow that there exists a T_o such that $n(t) \leq Ke^{r\tau}$ for $t \geq T_o$.

Let us now suppose n is oscillatory about K and let $n(t^*)$ denote an arbitrary local maximum of n; then

$$0 = \frac{dn(t^*)}{dt} \leq rn(t^*)\Big[1 - \frac{n(t^*-\tau)}{K}\Big] \tag{4.11}$$

and therefore

$$n(t^* - \tau) \leq K; \tag{4.12}$$

a consequence of (4.12) is that there exists $\xi \in [t^* - \tau, t^*]$ such that $n(\xi) = K$. Integrating the inequality (4.9) from ξ to t^*, we have

$$\ell n\frac{n(t^*)}{K} \le \int_{\xi}^{t^*} r\left[1-\frac{n(s-\tau)}{K}\right]ds$$
$$\le r\int_{\xi}^{t^*} ds \le r\int_{t^*-\tau}^{t^*} ds = r\tau$$

which implies that

$$n(t^*) \le Ke^{r\tau}.$$

Since $n(t^*)$ is an arbitrary local maximum of N, we can conclude that

$$n(t) \le n(t^*) \le Ke^{r\tau} = L \tag{4.13}$$

for $t \ge T_o$, where T_o is the first zero of the oscillatory n. We have from (4.13) and the second of (4.1) that

$$\frac{du(t)}{dt} \le -au(t) + bL \quad \text{for} \quad t \ge T_o + \tau. \tag{4.14}$$

We can compare solutions of (4.14) with those of

$$\frac{dx(t)}{dt} = -ax(t) + bL, \quad x(T_o+\tau) = u(T_o+\tau). \tag{4.15}$$

It is easy to see that solutions of (4.15) satisfy

$$x(t) \to \frac{bL}{a} \quad \text{as} \quad t \to \infty.$$

Thus by comparison of (4.14) and (4.15) we can conclude that

$$u(t) \le x(t) \qquad \text{for} \quad t \ge T_o + \tau.$$

Thus there exists a $T_1 \ge 0$ such that

$$u(t) \le \frac{bL}{a} + \epsilon = M \quad \text{for} \quad t \ge T_1. \tag{4.16}$$

The priori upper bounds of n and u in (4.7) follow from (4.13) and (4.16). Using these upper bounds of n and u one can derive the lower bounds in a similar way. We shall omit the details.

The next result provides verifiable sufficient conditions for the global asymptotic stability of the positive equilibrium (n^*, u^*) of (4.1).

Theorem 4.2. *Suppose that a, b, c, r, τ, K are as in Lemma 4.1. In addition suppose that the following holds:*

$$K\Big(\frac{bc}{a} + \frac{r\tau}{n^*}\Big)e^{r\tau} < \frac{1}{2}. \tag{4.17}$$

Then all the positive solutions $(n(t), u(t))$ of (4.1) satisfy

$$\lim_{t\to\infty}(n(t), u(t)) = (n^*, u^*). \tag{4.18}$$

Proof. Let $\Big(n(t), u(t)\Big)$ be any positive solution of (4.1). By Lemma 4.1,

$$\frac{dn(t)}{dt} \leq rn(t)\Big[1 - cm - \frac{n(t-\tau)}{K}\Big] \tag{4.19}$$

eventually and this will lead to the existence of $t_1' > 0$ such that

$$n(t) \leq (1 - cm)Kexp\Big\{\Big(1 - cm - \frac{l}{K}\Big)r\tau\Big\} = L^{(1)}, \quad \text{for} \quad t \geq t_1'; \tag{4.20}$$

details of proof are similar to our discussion in the proof of Lemma 4.1, where $1 - cm > 1 - cM = \alpha_1 > 0$, $t_1' > T^*$. Replacing (4.20) in the second equation of (4.1), we have

$$\frac{du(t)}{dt} \leq -au(t) + bL^{(1)} \qquad \text{eventually for large} \quad t. \tag{4.21}$$

As a consequence of (4.21) we obtain

$$u(t) \leq \frac{b}{a}L^{(1)} + \epsilon_1 = M^{(1)} \quad \text{for} \quad t \geq t_1'', \tag{4.22}$$

where $t_1'' \geq t_1'$, and

$$0 < \epsilon_1 \leq \min\Big\{\epsilon, \frac{1}{2}\Big\}. \tag{4.23}$$

Combining (4.22) and the first equation of (4.1), we have

$$\frac{dn(t)}{dt} \geq rn(t)\Big[1 - cM^{(1)} - \frac{n(t-\tau)}{K}\Big] \tag{4.24}$$

eventually and this implies

$$n(t) \geq (1 - cM^{(1)})Kexp\Big\{\Big(1 - cM^{(1)} - \frac{L^{(1)}}{K}\Big)r\tau\Big\} = \ell^{(1)} \quad \text{for} \quad t \geq t_1''', \tag{4.25}$$

where $t_1''' > t_1''$. Replacing (4.25) in the second equation of (4.1), we get

$$\frac{du(t)}{dt} \geq -au(t) + b\ell^{(1)} \tag{4.26}$$

eventually, which gives

$$u(t) \geq \frac{b}{a}\ell^{(1)} - \epsilon_1 = m^{(1)} \quad \text{for} \quad t \geq t_1, \tag{4.27}$$

where $t_1 \geq t_1'''$. Summarizing the above procedure, we know that for ϵ_1 satisfying (4.23), there is a $t_1 \geq T^*$ such that

$$\left.\begin{aligned} \ell^{(1)} \leq n(t) \leq L^{(1)} \\ m^{(1)} \leq u(t) \leq M^{(1)} \end{aligned}\right\} \quad \text{for} \quad t \geq t_1. \tag{4.28}$$

Let

$$L^{(0)} = L, \quad M^{(0)} = M, \quad \ell^{(0)} = \ell, \quad m^{(0)} = m, \quad \epsilon_0 = \epsilon,$$

then one can verify (for more details of the technique we refer to Gopalsamy [5] or Gopalsamy and Ahlip [6])

$$\left.\begin{aligned} \ell^{(0)} < \ell^{(1)} \leq n(t) \leq L^{(1)} < L^{(0)} \\ m^{(0)} < m^{(1)} \leq u(t) \leq M^{(1)} < M^{(0)} \end{aligned}\right\} \quad \text{for} \quad t \geq t_1.$$

Continuing the above procedure and using induction, we can derive the following relations among the various sequences:

$$\left.\begin{aligned}
&\{\epsilon_n\}; \quad \epsilon_n \le \min\left\{\frac{1}{n}, \epsilon_{n-1}\right\}, \quad \lim_{n\to\infty} \epsilon_n = 0;\\
&\{L^{(n)}\}; \quad L^{(n)} = (1 - cm^{(n-1)})Kexp\left\{\left(1 - cm^{(n-1)} - \frac{\ell^{(n-1)}}{K}\right)r\tau\right\};\\
&\{M^{(n)}\}; \quad M^{(n)} = \frac{b}{a}L^{(n)} + \epsilon_n;\\
&\{l^{(n)}\}; \quad l^{(n)} = (1 - cM^{(n)})Kexp\left\{\left(1 - cM^{(n)} - \frac{L^{(n)}}{K}\right)r\tau\right\};\\
&\{m^{(n)}\}; \quad m^{(n)} = \frac{b}{a}\ell^{(n)} - \epsilon_n;\\
&\{t_n\}; \quad t_n \ge t_{n-1}, \quad \lim_{n\to\infty} t_n = \infty;
\end{aligned}\right\} \tag{4.29}$$

and the above sequences satisfy:

$$\left.\begin{aligned}
\ell^{(n-1)} < \ell^{(n)} \le n(t) \le L^{(n)} < L^{(n-1)}\\
m^{(n-1)} < m^{(n)} \le u(t) \le M^{(n)} < M^{(n-1)}
\end{aligned}\right\} \quad \text{for} \quad t \ge t_n. \tag{4.30}$$

(4.29) and (4.30) together imply that the respective sequences converge to finite nonnegative limits as $n \to \infty$:

$$\left.\begin{aligned}
&\lim_{n\to\infty} L^{(n)} = L^*, \quad \lim_{n\to\infty} M^{(n)} = M^*, \quad \lim_{n\to\infty} \ell^{(n)} = \ell^*, \quad \lim_{n\to\infty} m^{(n)} = m^*,\\
&l^* \le n^* \le L^*, \quad m^* \le u^* \le M^*,
\end{aligned}\right\} \tag{4.31}$$

and

$$\left.\begin{aligned}
L^* &= (1 - cm^*)Kexp\left\{\left(1 - cm^* - \frac{\ell^*}{K}\right)r\tau\right\}\\
l^* &= (1 - cM^*)Kexp\left\{\left(1 - cM^* - \frac{L^*}{K}\right)r\tau\right\}\\
M^* &= \frac{b}{a}L^*, \quad m^* = \frac{b}{a}\ell^*.
\end{aligned}\right\} \tag{4.32}$$

We note

$$\frac{bc}{a} + \frac{1}{K} = \frac{1}{n^*} \tag{4.33}$$

and derive from (4.32) that

$$\begin{aligned}
L^* =& K\left(1 - \frac{bc\ell^*}{a}\right)exp\left\{\left(1 - \frac{bc\ell^*}{a} - \frac{\ell^*}{K}\right)r\tau\right\}\\
=& K\left\{1 - \frac{bc}{a}(1 - cM^*)Kexp\left[\left(1 - cM^* - \frac{L^*}{K}\right)r\tau\right]\right\}\\
&exp\left\{\left\{1 - \left(\frac{bc}{a} + \frac{1}{K}\right)(1 - cM^*)Kexp\left[\left(1 - cM^* - \frac{L^*}{K}\right)r\tau\right]\right\}r\tau\right\}\\
=& K\left\{1 - \frac{bcK}{a}(1 - cM^*)exp\left[\left(1 - cM^* - \frac{L^*}{K}\right)r\tau\right]\right\}\\
&exp\left\{\left\{1 - \frac{K}{n^*}(1 - cM^*)exp\left[\left(1 - cM^* - \frac{L^*}{K}\right)\right]r\tau\right\}r\tau\right\}\\
=& K\left\{1 - \frac{bcK}{a}\left(1 - \frac{bcL^*}{a}\right)exp\left[\left(1 - \frac{L^*}{n^*}\right)r\tau\right]\right\}\\
&exp\left\{\left\{1 - \frac{K}{n^*}\left(1 - \frac{bcL^*}{a}\right)exp\left[\left(1 - \frac{L^*}{n^*}\right)r\tau\right]\right\}r\tau\right\}.
\end{aligned} \tag{4.34}$$

Similarly we have

$$\begin{aligned}\ell^* =& K\Big(1-\frac{bcL^*}{a}\Big)exp\Big\{\Big(1-\frac{bcL^*}{a}-\frac{L^*}{K}\Big)r\tau\Big\}\\ =& K\Big\{1-\frac{bcK}{a}\Big(1-\frac{bc\ell^*}{a}\Big)exp\Big[\Big(1-\frac{\ell^*}{n^*}\Big)r\tau\Big]\Big\}\\ & exp\Big\{\Big\{1-\frac{K}{n^*}\Big(1-\frac{bc\ell^*}{a}\Big)exp\Big[\Big(1-\frac{\ell^*}{n^*}\Big)r\tau\Big]\Big\}r\tau\Big\}.\end{aligned} \tag{4.35}$$

The proof will be complete if we can show that

$$l^* = n^* = L^*.$$

Define functions h, g and f as follows:

$$\begin{aligned}&h(x) = exp\Big[\Big(1-\frac{x}{n^*}\Big)r\tau\Big], \quad g(x) = \Big(1-\frac{bc}{a}x\Big)h(x),\\ &f(x) = K\Big[1-\frac{bcK}{a}g(x)\Big]exp\Big\{\Big[1-\frac{K}{n^*}g(x)\Big]r\tau\Big\}.\end{aligned} \tag{4.36}$$

The following identity

$$n^* = K\Big\{1-\frac{bcK}{a}\Big(1-\frac{bc}{a}n^*\Big)\Big\}, \tag{4.37}$$

is a consequence of (4.33); we have from (4.34)-(4.37) that ℓ^*, n^*, L^* are roots of the functional equation

$$x = f(x). \tag{4.38}$$

Note that $1 - cM^* = 1 - \frac{bc}{a}L^* > 1 - \frac{bcK}{a}e^{r\tau} > 0$, which in combination with (4.30)-(4.31) leads to: $0 < \ell^* \le n^* \le L^* < \frac{a}{bc}$. It can be found from our hypotheses that $f(0) = K\Big(1 - \frac{bcK}{a}e^{r\tau}\Big)exp\Big\{\Big(1-\frac{K}{n^*}e^{r\tau}\Big)r\tau\Big\} > 0$. If we can show that $f''(x) < 0$, for $x \in [0, \frac{a}{bc})$; $(' = \frac{d}{dx})$ (i.e. $f(x)$ is an upper convex function in the interval $I = [0, \frac{a}{bc})$), then we know (4.38) has a unique solution in I from which it will follow that $L^* = n^* = \ell^*$ immediately. Now we calculate the derivatives of $f(x)$ for $x \in I$. Define g_1 and g_2 as follows:

$$\left.\begin{aligned}&g_1(x) = K\Big[1-\frac{bcK}{a}g(x)\Big] > 0\\ &g_2(x) = exp\Big\{\Big[1-\frac{K}{n^*}g(x)\Big]r\tau\Big\} > 0\end{aligned}\right\} \quad \text{for} \quad x \in I. \tag{4.39}$$

We verify by direct calculation that for $x \in I$,

$$\left.\begin{aligned}g'(x) &= \Big[-\frac{bc}{a}-\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]h(x) < 0\\ g''(x) &= \Big[\frac{2r\tau}{n^*}\cdot\frac{bc}{a}+\Big(\frac{r\tau}{n^*}\Big)^2\Big(1-\frac{bc}{a}x\Big)\Big]h(x) > 0\\ g_1'(x) &= -\frac{bcK^2}{a}g'(x)\\ g_1''(x) &= -\frac{bcK^2}{a}g''(x)\\ g_2'(x) &= -\frac{Kr\tau}{n^*}g'(x)exp\Big\{\Big[1-\frac{K}{n^*}g(x)\Big]r\tau\Big\} = -\frac{Kr\tau}{n^*}g'(x)g_2(x)\\ g_2''(x) &= -\frac{Kr\tau}{n^*}g_2(x)g''(x)+\Big[\frac{Kr\tau}{n^*}g'(x)\Big]^2g_2(x)\\ f(x) &= g_1(x)g_2(x)\\ f''(x) &= g_1(x)g_2''(x)+g_2(x)g_1''(x)+2g_1'(x)g_2'(x)\\ &= \frac{Kr\tau}{n^*}g_1(x)g_2(x)\Big\{-g''(x)+\frac{Kr\tau}{n^*}[g'(x)]^2\Big\}\\ &\quad+\frac{bcK^2}{a}g_2(x)\Big\{-g''(x)+\frac{2Kr\tau}{n^*}[g'(x)]^2\Big\}.\end{aligned}\right\} \tag{4.40}$$

But (4.17) implies

$$\begin{aligned}2K\Big[\frac{bc}{a}+\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]h(x) =&2K\Big[\frac{bc}{a}+\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]exp\Big[\Big(1-\frac{x}{n^*}\Big)r\tau\Big]\\ \le&2K\Big(\frac{bc}{a}+\frac{r\tau}{n^*}\Big)e^{r\tau}<1 \quad \text{for} \quad x\in I \end{aligned} \tag{4.41}$$

and this leads to

$$\begin{aligned}2K\Big[\frac{bc}{a}+\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]^2h^2(x) <&\Big[\frac{bc}{a}+\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]h(x)\\ <&\Big[\frac{2bc}{a}+\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]h(x) \qquad \text{for} \quad x\in I. \end{aligned} \tag{4.42}$$

Using (4.42) we derive that

$$\begin{aligned}-g''(x)+\frac{2Kr\tau}{n^*}[g'(x)]^2 =&-\Big[\frac{2r\tau}{n^*}\cdot\frac{bc}{a}+\Big(\frac{r\tau}{n^*}\Big)^2\Big(1-\frac{bc}{a}x\Big)\Big]h(x)\\ &+\frac{2Kr\tau}{n^*}\Big[-\frac{bc}{a}-\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]^2h^2(x)\\ =&\frac{r\tau}{n^*}\Big\{2K\Big[\frac{bc}{a}+\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]^2h^2(x)\\ &-\Big[\frac{2bc}{a}+\frac{r\tau}{n^*}\Big(1-\frac{bc}{a}x\Big)\Big]h(x)\Big\}\\ <&0 \qquad \text{for} \quad x\in I, \end{aligned} \tag{4.43}$$

which gives

$$-g''(x)+\frac{Kr\tau}{n^*}[g'(x)]^2<0 \qquad \text{for} \quad x\in I. \tag{4.44}$$

We can now conclude from (4.39), (4.40), (4.43) and (4.44) that

$$f''(x)<0 \quad \text{for} \quad x\in I. \tag{4.45}$$

It is now a consequence of (4.45) that (4.38) has a unique positive root. Thus $\ell^*=n^*=L^*$ and hence $M^*=u^*=m^*$. Combining (4.30) and (4.31), we obtain

$$\lim_{t\to\infty}\big(N(t),u(t)\big)=\big(n^*,u^*\big)$$

and this completes the proof.

As a special case if we let $c=0$ in the above analysis then have

$$\begin{aligned} n^*=K,\quad & g(x)=exp\Big\{\Big(1-\frac{x}{K}\Big)r\tau\Big\},\\ g_1(x)=K,\quad & g_2(x)=exp\Big\{\Big[1-e^{(1-\frac{x}{K})r\tau}\Big]r\tau\Big\},\\ f''(x)=Kr\tau g_2(x)&\Big\{-g''(x)+r\tau[g'(x)]^2\Big\}. \end{aligned}$$

If we assume furthermore that the delay is small enough to satisfy

$$r\tau e^{r\tau}<1, \tag{4.46}$$

then we could conclude that

$$-g''(x) + r\tau[g'(x)]^2 < 0 \qquad \text{for} \quad x \geq 0,$$

and this will imply that

$$f''(x) < 0 \qquad \text{for} \quad x \geq 0.$$

It will again follow from $f(0) > 0$ and the above that $f(x) = x$ has a unique positive root leading to

$$\lim_{t\to\infty} n(t) = K = n^*. \tag{4.47}$$

This corresponds to the global asymptotic stability of the positive equilibrium of the uncontrolled delay equation:

$$\frac{dn(t)}{dt} = rn(t)\Big[1 - \frac{n(t-\tau)}{K}\Big]. \tag{4.48}$$

The sufficient condition in (4.46) for (4.47) to hold in the case of (4.48) is the same as that obtained by Gopalsamy [7] using Lyapunov-function type arguments.

The authors believe that the sufficient condition of Theorem can be improved; in particular the authors conjecture that the conclusion of Theorem 4.2 holds when any one of the following inequalities is satisfied:

$$K\left(\frac{bc}{a} + \frac{r\tau}{n^*}\right)e^{r\tau} < 1 \qquad \text{or}$$
$$K\left(\frac{bc}{a} + \frac{r\tau}{n^*}\right) < 1.$$

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