A CHARACTERIZATION OF FUZZY NEIGHBORHOOD COMMUTATIVE DIVISION RINGS

T.M.G. AHSANULLAH and FAWZI AL-THUKAIR

Department of Mathematics, College of Science
King Saud University
P.O. Box 2455, Riyadh-11451
Kingdom of Saudi Arabia

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ABSTRACT. We give a characterization of fuzzy neighborhood commutative division ring; and present an alternative formulation of boundedness introduced in fuzzy neighborhood rings. The notion of $\beta$-restricted fuzzy set is considered.

KEY WORDS AND PHRASES. Fuzzy neighborhood system; fuzzy neighborhood commutative division ring (FNCDR); bounded fuzzy set; $\beta$-restricted fuzzy set.

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1. INTRODUCTION.

The notions of fuzzy neighborhood division ring and fuzzy neighborhood commutative division ring are announced in [1] without producing any characterization theorem on the topics. In this article, our aim is to provide with such a characterization theorem.

Fuzzy neighborhood rings are studied in [2] where the concept of bounded fuzzy set is introduced. We give here an alternative equivalent formulation of boundedness in case of commutative division rings. Finally, we propose a notion of $\beta$-restricted fuzzy set where $0 < \beta \leq 1$, an analogue of restricted set in topological commutative division rings.

2. PRELIMINARIES.

Like recent works, for instance ([1], [2], [3], [4] and [5]) the key item of this article is the notion of fuzzy neighborhood system originated by R. Lowen [6]. For our convenience, we quote below a few known definitions and useful results.

Throughout the text, we consider the triplet $(D, +, \cdot)$ either a ring, division ring or commutative division ring (whichever we require), while $D^* = D \setminus \{0\}$ stands for multiplicative group of nonzero elements of commutative division ring $D$ and $D^+$ is the additive group of $D$.

As usual, $I_0 = \{0,1\}$, and $I = \{0,1\}$ the unit interval. $\Box$ denotes the completion of the proof. For any fuzzy set $\mu I D(=\{\mu: D\rightarrow [0,1]\})\mu^\sim$ is defined as

$$\mu^\sim(x) = \mu(x^{-1}) \forall x \in D^*$$

If $x \in D$ then,

$$x \oplus \mu(y) = 1_{\{x\}} \oplus \mu(y) = \mu(y - x) \forall y \in D$$

where $1_{\{x\}}$ denotes the characteristic function of the singleton set $\{x\}$, while for any $\mu, \nu_1, \nu_2 I D$

and $x \in D^*$
\[ x \otimes \mu, \nu_1 \oplus \nu_2 \text{ and } \nu_1 \odot \nu_2 \text{ are defined successively,} \]

\[ x \otimes \mu(y) = 1_x \otimes \mu(y) = \mu(y/x^{-1}) \]

\[ \nu_1 \oplus \nu_2(y) = \bigvee_{s,t} \nu_1(s) \wedge \nu_2(t) \]

and \[ \nu_1 \odot \nu_2(y) = \bigvee_{s,t} \nu_1(s) \wedge \nu_2(t) \]

for all \( y \in D \).

Also, we define \( \mu/\nu \) as

\[ \nu/\nu = \mu \odot \nu \]

and so \( 1/(1 \oplus \nu) \) is written as

\[ 1/(1 \oplus \nu)(x) = (1 \oplus \nu)^{-1}(x) = (1 \oplus \nu)(x^{-1}) \quad \forall x \in D^*. \]

We call \( \mu \) is symmetric if and only if

\[ \mu = -\mu, \text{ where } \sim \mu(x) = \mu(-x) \forall x \in D. \]

The constant fuzzy set of \( D \) with value \( \delta \in I \) is given by the symbol \( D_\delta \).

We recall the so-called saturation operator \([6,7]\) which is defined on a prefilter base \( F \subset I \) by

\[ \bar{F} = \{ \nu \in I \mid \forall \delta \in I_0 \exists \nu \in F \ni \nu \sim -\delta \leq \nu \}. \]

If \( \Sigma = (\Sigma(x))_{x \in D} \) is a fuzzy neighborhood system on a set \( D \) then \( t(\Sigma) \) is the fuzzy neighborhood topology on \( D \), and the pair \((D, t(\Sigma))\) is known as fuzzy neighborhood space \([6]\).

**Proposition 2.1.** If \((D, t(\Sigma))\) and \((D', t'(\Sigma'))\) are fuzzy neighborhood spaces and \( f : D \rightarrow D' \), then \( f \) is continuous at \( x \in D \Leftrightarrow \forall \mu \in \Sigma'(f(x)) \quad \forall \delta \in I_0 \exists \nu \in \Sigma(x) \) such that \( \nu - \delta \leq f^{-1}(\nu) \).

**Definition 2.2.** Let \((D, +, \cdot)\) be a ring and \( \Sigma \) a fuzzy neighborhood system on \( D \). Then the quadruple \((D, +, \cdot, t(\Sigma))\) is said to be a fuzzy neighborhood ring if and only if the following are satisfied:

(FR1) The mapping \( h : (D \times D, t(\Sigma) \times t(\Sigma)) \rightarrow (D, t(\Sigma)), (x, y) \mapsto x + y \) is continuous.

(FR2) The mapping \( k : (D, t(\Sigma)) \rightarrow (D, t(\Sigma)), x \mapsto -x \) is continuous.

(FR3) The mapping \( m : (D \times D, t(\Sigma) \times t(\Sigma)) \rightarrow (D, t(\Sigma)), (x, y) \mapsto xy \) is continuous.

**Proposition 2.3.** Let \((D, +, \cdot, t(\Sigma))\) be a fuzzy neighborhood ring and \( x \in D \).

Then

(a) The left homothety \( L_x : (D, t(\Sigma)) \rightarrow (D, t(\Sigma)), y \mapsto xy \) (resp. right homothety \( R_x : (D, t(\Sigma)) \rightarrow (D, t(\Sigma)), y \mapsto yx \)) is continuous. If \( x \) is a unit element of \( D \) then each homothety is a homeomorphism.

(b) The translation \( T_x : (D, t(\Sigma)) \rightarrow (D, t(\Sigma)), y \mapsto y + x \), and the inversion \( k \) are homeomorphisms.

(c) \( \nu \in \Sigma(x) \Leftrightarrow x \oplus \nu \in \Sigma(x) \), i.e., \( T_x(\nu) \in \Sigma(x) \).

(d) \( \nu \in \Sigma(x) \Leftrightarrow -x \oplus \nu \in \Sigma(0) \), i.e., \( T_{-x}(\nu) \in \Sigma(0) \).

**Definition 2.4.** Let \((D, +, \cdot, t(\Sigma))\) be a division ring, and \( \Sigma \) a fuzzy neighborhood system on \( D \). Then the quadruple \((D, +, \cdot, t(\Sigma))\) is said to be a fuzzy neighborhood division ring if and only if the following are true:
(FD1) $(D, +, -, t(\Sigma))$ is a fuzzy neighborhood ring.

(FD2) The mapping $r: (D^*, (\Sigma | D^*)) \to (D^*, (\Sigma | D^*))$, $x \mapsto x^{-1}$ is continuous, where $\Sigma | D^*$ is the fuzzy neighborhood system on $D^*$ induced by $D$.

**THEOREM 2.5.** Let $(D, +, -)$ be a ring and $\Sigma$ a fuzzy neighborhood system on $D$. Then the quadruple $(D, +, -, t(\Sigma))$ is a fuzzy neighborhood ring if and only if the following are satisfied:

1. $\forall x \in D: \Sigma(x) = \{ T_x(\nu) : \nu \in \Sigma(0) \}$
2. $\forall x \in D, \forall \nu \in \Sigma(0), \exists \delta > 0$ such that $x \in \delta + \Sigma(0)$ and $\nu \in \delta + \Sigma(0)$, i.e., the mapping $y \mapsto x \circ y$ and $y \mapsto y \circ x$ are continuous at $0$.
3. $\forall \nu \in \Sigma(0), \forall \delta > 0, \exists \delta > 0$ such that $x \in \delta + \Sigma(0)$ and $\nu \in \delta + \Sigma(0)$, i.e., the mapping $(x, y) \mapsto x \circ y$ is continuous at $(0, 0)$.
4. $\forall \nu \in \Sigma(0), \forall \delta > 0, \exists \delta > 0$ such that $x \in \delta + \Sigma(0)$ and $\nu \in \delta + \Sigma(0)$, i.e., the mapping $(x, y) \mapsto x \circ y$ is continuous at $(0, 0)$.

**3. CHARACTERIZATION OF FNCDR AND SOME OTHER RESULTS.**

The following is a characterization of fuzzy neighborhood commutative division ring. We consider $\Sigma(0)$ to be symmetric fuzzy neighborhoods of zero.

**THEOREM 3.1.** Let $(D, +, -)$ be a commutative division ring and $(D, +, -, t(\Sigma))$ a fuzzy neighborhood ring. Then the quadruple $(D, +, -, t(\Sigma))$ is a fuzzy neighborhood commutative division ring if and only if the following are fulfilled:

1. $\forall x \in D: \Sigma(x) = \{ T_x(\nu) = x \circ \nu : \nu \in \Sigma(0) \}$.
2. $\forall \mu \in \Sigma(0), \forall \nu \in \Sigma(0), \exists \delta > 0$ such that $x \in \delta + \Sigma(0)$ and $\nu \in \delta + \Sigma(0)$, i.e., the mapping $y \mapsto x \circ y$ and $y \mapsto y \circ x$ are continuous at $0$.
3. $\forall \nu \in \Sigma(0), \forall \delta > 0, \exists \delta > 0$ such that $x \in \delta + \Sigma(0)$ and $\nu \in \delta + \Sigma(0)$, i.e., the mapping $(x, y) \mapsto x \circ y$ is continuous at $(0, 0)$.
4. $\forall \mu \in \Sigma(0), \forall \nu \in \Sigma(0), \exists \delta > 0$ such that $x \in \delta + \Sigma(0)$ and $\nu \in \delta + \Sigma(0)$, i.e., the mapping $(x, y) \mapsto x \circ y$ is continuous at $(0, 0)$.
5. $\forall \nu \in \Sigma(0), \forall \delta > 0, \exists \delta > 0$ such that $x \in \delta + \Sigma(0)$ and $\nu \in \delta + \Sigma(0)$, i.e., the inversion $x \mapsto x^{-1}$ is continuous at $1$.

**PROOF.** If $(D, +, -, t(\Sigma))$ is a fuzzy neighborhood commutative division ring, then the conditions (i) - (iv) are immediate from Theorem 2.5. We check condition (v).

Let $\mu \in \Sigma(0)$ and $\delta \in \Sigma(0)$, then $1 \oplus \mu \in \Sigma(0)$. Since $r: x \mapsto x^{-1}$ is continuous at $1$, we can find $\nu \in \Sigma(0)$ such that $1 \oplus \nu \in \Sigma(0)$ and $r(1 \oplus \nu) \leq (1 \oplus \mu) + \beta$.

But $r(1 \oplus \nu) \leq (1 \oplus \nu) \leq (1 \oplus \mu) + \beta$.

Conversely, if the conditions (i) - (v) are fulfilled then only we need to prove that the inversion $r: x \mapsto x^{-1}$ is continuous, i.e., we show that

$$\forall \nu \in \Sigma(0), \forall x \in D^*, \forall \delta > 0, \exists \delta > 0$$

$$x \mapsto (x \oplus \nu) \leq (x \oplus \mu) + \delta.$$  

Let $x$ be an element of $D^*$, and $\delta \in \Sigma(0)$. Then in view of (ii), there is a $\mu \in \Sigma(0)$ such that

$$\mu \circ x \leq (1 \oplus \mu) + \beta.$$  

Now due to (v), corresponding to $\mu$ we can find $\nu \in \Sigma(0)$ such that

$$\nu \in (1 \oplus \nu) \leq (1 \oplus \mu) + \beta.$$  

Then by (ii), there exists a $\nu \in \Sigma(0)$ such that

$$x \circ (1 \oplus \nu) \leq (x \circ (1 \oplus \mu) + \beta.$$  

Now
But then with simplification, we have
\[ (x \oplus \nu) \sim = x \sim \circ (1 \oplus (x \sim \circ \nu)) \sim \leq (x \sim \oplus \mu) + \frac{\delta}{3}, \]
which proves (*).

**PROPOSITION 3.2.** Let \((D, +, \cdot, t(\Sigma))\) be a fuzzy neighborhood commutative division ring. If the conditions (i) - (v) of Theorem 3.1 are satisfied then the following inequality hold good.

\[ \forall \mu \Sigma(0), \forall \delta I \exists \nu \Sigma(0) \exists \nu/(1 \oplus \nu) \leq \mu + \frac{\delta}{3}. \]

**PROOF.** Suppose that the condition (i) - (v) hold good. Let \(\mu \Sigma(0)\) and \(\delta I_0\). Then there are \(\mu_1, \mu_2 \Sigma(0)\) such that
\[ \mu_1 \oplus \mu_1 \leq \mu + \frac{\delta}{3}, \mu_2 \leq \mu_1; \]
and
\[ \mu_2 \oplus \mu_2 \leq \mu_1 + \frac{\delta}{3}. \tag{3.4} \]
By (v), for every \(\mu_2 \Sigma(0) \exists \nu \Sigma(0), \nu \leq \mu_2\) such that
\[ (1 \oplus \nu) \sim \leq (1 \oplus \mu_2) + \frac{\delta}{3}. \tag{3.5} \]
Then we have
\[
\nu/(1 \oplus \nu) = \nu \circ (1 \oplus \nu) \sim (by \ definition) \\
\leq \nu \circ (1 \oplus \mu_2) + \frac{\delta}{3} \leq (\nu \circ 1) \circ (\nu \circ \mu_2) + \frac{\delta}{3} \quad (by \ (3.5)) \\
\leq \mu_2 \oplus (\mu_2 \oplus \mu_2) + \frac{\delta}{3} \\
\leq (\mu_2 \oplus \mu_1) + \frac{\delta}{3} \\
\leq (\mu_1 \oplus \mu_1) + \frac{\delta}{3} \\
\leq \mu + \frac{\delta}{3} \\
\Rightarrow \nu/(1 \oplus \nu) \leq \mu + \frac{\delta}{3}. \]

**THEOREM 3.3.** Let \((D, +, \cdot)\) be a commutative division ring equipped with a fuzzy neighborhood topology \(t(\Sigma)\). If \((D, +, t(\Sigma))\) is a fuzzy neighborhood group with respect to addition
\[ h:(D \times D, t(\Sigma) \times t(\Sigma)) \rightarrow (D, t(\Sigma)), (x, y) \mapsto x + y \]
and \((D^*, \cdot, t(\Sigma))\) is a fuzzy neighborhood group with respect to multiplication
\[ m:(D^* \times D^*, t(\Sigma) \times t(\Sigma)) \rightarrow (D^*, t(\Sigma)), (x, y) \mapsto xy, \]
then \((D, +, \cdot, t(\Sigma))\) is a fuzzy neighborhood commutative division ring.

**PROOF.** As the inversion, the addition and subtraction, i.e.,
\[ r:(D^*, t(\Sigma)) \rightarrow (D, t(\Sigma)), x \mapsto -1, \]
\[ h:(D \times D, t(\Sigma) \times t(\Sigma)) \rightarrow (D, t(\Sigma)), (x, y) \mapsto x + y, \]
\[ h':(D \times D, t(\Sigma) \times t(\Sigma)) \rightarrow (D, t(\Sigma)), (x, y) \mapsto x - y, \]
are continuous, it is sufficient to show that the multiplication \( m: (D \times D, t(\Sigma) \times t(\Sigma)) \to (D, t(\Sigma)), (x, y) \mapsto xy \) is continuous.

Let \( \Sigma(0) \) be symmetric fuzzy neighborhoods of zero in the additive group \( D^+ \) of \( D \), and

\[
\Sigma(x) = \{ x \oplus \nu \mid \nu \in \Sigma(0) \}
\]

We show that

\[
\forall x \in D, \forall y \in D, \forall \mu \in \Sigma(0), \forall \delta \in I_0 \exists \nu \in \Sigma(0) \exists \eta \in \Sigma(0) \exists \theta \in \Sigma(0) \exists \theta' \in \Sigma(0)

(\nu \oplus x) \circ (\nu \oplus y) - \eta \leq \mu \oplus xy.
\]

Condition (***) is satisfied for all \( x \in D^*, y \in D^* \).

Indeed,

\[
\forall x, y \in D^*, \forall \mu \in \Sigma(0), \forall \delta \in I_0 \exists \nu \in \Sigma(0) \exists \eta \in \Sigma(0)

\theta \circ \theta - \delta \leq \mu_x y.
\]

We let \( \mu_{xy} = \mu \oplus xy \) with \( \mu \in \Sigma(0) \),

\[
\theta_x = x \oplus \theta \epsilon \Sigma(x) \theta_y = y \oplus \theta' \epsilon \Sigma(y) \text{ with } \theta, \theta' \epsilon \Sigma(0).
\]

Put \( \nu = \theta \wedge \theta' \). Then we have

\[
(\nu \oplus x) \circ (\nu \oplus y) - \delta \leq (\theta \oplus x) \circ (\theta' \oplus x) - \delta
\]

\[
\leq \theta \circ \theta - \delta \leq \mu_{xy} = \mu \oplus xy,
\]

as desired.

It remains to show that if \( xy = 0 \), then (***) is satisfied. First, let \( x = y = 0 \); suppose \( \mu \in \Sigma(0) \) and \( \delta \in I_0 \); then by Proposition 2.3(c), \( 1 \oplus \mu \Sigma(1) \).

There exists \( \theta \in \Sigma(0) \) such that

\[
\theta \oplus \theta \oplus \theta \leq \mu + \delta/2.
\]

Consequently, as multiplication \( m: (x, y) \mapsto xy \) is continuous at \( (1, 1) \), there exists \( \nu_1 \in \Sigma(1) \) such that

\[
\nu_1 \oplus \nu_1 \leq (1 \oplus \theta) + \delta/2.
\]

Then in view of Proposition 2.3(d), \( -1 \oplus \nu_1 \epsilon \Sigma(0) \). Let us put

\[
\nu = -1 \oplus \nu_1 \text{ and } \nu = \nu \wedge \theta
\]

then \( \nu \in \Sigma(0) \) and hence,

\[
\nu \circ \nu = (-1 \oplus \nu_1) \circ (-1 \oplus \nu_1)
\]

\[
\leq 1 \oplus ((-1) \oplus \nu_1) \oplus (\nu_1 \oplus (-1)) \oplus (\nu_1 \circ \nu_1)
\]

\[
\leq (1 \oplus \sim \nu_1) \oplus (\sim \nu_1) \oplus (1 \oplus \theta) + \delta/2
\]

\[
\leq \theta \oplus \theta \oplus \theta + \delta/2 \leq \mu + \delta
\]

which proves that

\[
\forall \mu \in \Sigma(0), \forall \delta \in I_0 \exists \nu \in \Sigma(0) \exists \nu \leq \mu + \delta.
\]
Next, let $x \neq 0 = y$. Since the multiplication $m(x,y) = xy$ is continuous at $(1, x)$, then (****) implies that

$$\forall \mu(x) \in \Sigma(x), \forall \nu \in \Sigma(1) \exists \nu' \in \Sigma(x) \exists \nu_1 \in \Sigma \nu_1 \circ \nu_x - \frac{\delta}{3} \leq \mu_x.$$  \hfill (3.8)

Choose

$$\mu_x = \mu \oplus x, \text{ with } \mu \in \Sigma(0);$$
$$\nu_1 = 1 \oplus \theta, \nu_x = x \oplus \theta' \text{ with } \theta, \theta' \in \Sigma(0).$$

Set $v = \theta \land \theta'$. Then it follows immediately that

$$(1 \oplus v) \circ (x \oplus v) \leq (\mu \oplus x) + \frac{\delta}{3}.$$  \hfill (by (3.8))

and consequently, $(v \oplus x) \circ v \leq \mu + \frac{\delta}{3}$ which proves that

$$\forall \mu(x) \in \Sigma(x), \forall \nu \in \Sigma(1) \exists \nu_1 \in \Sigma(x) \exists \nu_1 \circ \nu_x - \frac{\delta}{3} \leq \mu_x.$$

**DEFINITION 3.4.** Let $(D, +, \cdot)$ be a commutative division ring and $(D, +, \cdot, t(\Sigma))$ a fuzzy neighborhood ring. Then a fuzzy set $\mu \in D$ is said to be bounded in $(D, +, \cdot, t(\Sigma))$ if and only if for all $\nu \in \Sigma(0)$ and all $\delta > 0$ there exists $\theta \in \Sigma(0)$ such that $\mu \circ \theta \leq \nu + \frac{\delta}{3}$.

**PROPOSITION 3.5.** Let $(D, +, \cdot)$ be a commutative division ring and $(D, +, \cdot, t(\Sigma))$ a fuzzy neighborhood ring. Then the following statements are equivalent:

(B1): $\mu \in D$ is bounded in $(D, +, \cdot, \Sigma(\Sigma))$;

(B2): $\forall \mu \in \Sigma(0), \forall \nu \in \Sigma(1) \exists \mu \circ x \leq \nu + \frac{\delta}{3}$.

**PROOF.** (B1) $\Rightarrow$ (B2) is trivial, we prove (B2) $\Rightarrow$ (B1). Let $\mu \in D$, $\nu \in \Sigma(0)$ and $\delta > 0$. Then in view of Theorem 3.1 (iv) there exists a $\nu' \in \Sigma(0)$ such that

$$\nu' \circ \nu' - \delta/3 \leq \nu$$

By hypothesis, there is $\exists x \in D$ such that

$$\mu \circ x - \delta/3 \leq \nu'$$

Thus we have

$$\nu' \circ (\mu \circ x) \leq \nu' \circ \nu' + \delta/3$$

(by (3.10))

$$\leq \nu' + \delta/3$$

(by (3.9))

Again applying Theorem 3.1(ii), we can find $\theta \in \Sigma(0)$ such that

$$\theta \circ x \leq \nu' + \delta/3$$

$$\Rightarrow \theta \leq \nu' \circ x + \delta/3$$

(3.12)

So for any $x \in D$:

$$\mu \circ \theta(x) = \bigvee_{x \in D} \mu(x) \land \theta(t)$$

$$\leq \bigvee_{x \in D} \mu(x) \land (\nu' \circ x)(t) + \delta/3$$

$$= \mu \circ (\nu' \circ x)(z) + \delta/3$$

$$\leq \nu(z) + \delta/3 + \delta/3 = \nu(z) + \delta$$

(by (3.11))

$$\Rightarrow \mu \circ \theta \leq \nu + \delta.$$

\hfill $\square$
DEFINITION 3.6. Let \((D, +, \cdot)\) be a commutative division ring and \((D, +, \cdot, t(\Sigma))\) a fuzzy neighborhood ring. A fuzzy set \(\mu I^D\) is said to be \(\beta\)-restricted in \((D, +, \cdot, t(\Sigma))\) for \(0 < \beta \leq 1\) if and only if
\[
\overline{\mu}(0) < \beta.
\]
Where \(\overline{\mu}\) is the fuzzy closure operator given in Proposition 2.3 [6]

PROPOSITION 3.7. Let \((D, +, \cdot)\) be a division ring and \((D, +, \cdot, t(\Sigma))\) a fuzzy neighborhood ring. Then the following statements are equivalent:

(R1): \(\mu I^D\) is \(\beta\)-restricted in \((D, +, \cdot, t(\Sigma))\) for \(0 < \beta \leq 1\);

(R2): \(\exists \nu \in \Sigma(0) \exists \mu \in \nu(1) < \beta\).

PROOF. (R1) \(\Rightarrow\) (R2). Let \(0 < \beta \leq 1\), and \(\mu I^D\) be \(\beta\)-restricted. Suppose that \(\nu \in \Sigma(0)\) is such that \(\mu \in \nu(1) \geq \beta\); i.e., \(\forall y \in D^*\) such that \(y = y^{-1}\) such that
\[
\nu(y) \geq \beta
\]
\[
\Rightarrow \exists \nu \in \Sigma(0) \exists \nu \in \nu(1) \exists \mu \in \nu(1) < \beta.
\]

Now we have
\[
\nu \in \Sigma(0) \forall y \in D^* \mu \sim (y) \land \nu(y) \geq \beta.
\]
\[
\Rightarrow \nu \in \Sigma(0) \forall y \in D^* \mu \sim (y) \land \nu(y) \geq \beta.
\]
\[
\Rightarrow \nu \in \Sigma(0) \forall y \in D^* \mu \sim (y) \land \nu(y) \geq \beta.
\]
\[
\Rightarrow \mu \sim (y) \land \nu(y) \geq \beta,
\]
contradiction with the fact that \(\mu\) is \(\beta\)-restricted. (R2) \(\Rightarrow\) (R1). Let \(\mu I^D\) be not \(\beta\)-restricted for \(0 < \beta \leq 1\).

This means simply that
\[
\nu \in \Sigma(0) \forall y \in D^* \mu \sim (y) \land \nu(y) \geq \beta
\]
\[
\Rightarrow \nu \in \Sigma(0) \forall y \in D^* \mu \sim (y) \land \nu(y) \geq \beta.
\]
Now we have
\[
\mu \in \nu(1) = \bigvee_{s=1}^t \mu(s) \land \nu(t)
\]
\[
= \bigvee_{s=1}^t \mu \sim (s) \land \nu(t) \geq \beta
\]
a contradiction with (R2).

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