DIVERGENT SEQUENCES SATISFYING THE LINEAR FRACTIONAL TRANSFORMATIONS

A. McD. MERCER

Department of Mathematics and Statistics
University of Guelph
Ontario, N1G 2W1, Canada

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ABSTRACT. A real sequence $\{x_n\}_1^\infty$ which satisfies the recurrence $x_{n+1} = \frac{ax_n + b}{cx_n + d}$, in which all of a, b, c, d are real will, for certain values of these constants, be divergent. It is the purpose of this note to examine the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x_n); f \in C(-\infty, \infty)$$

in these cases. Except for certain exceptional values of a, b, c, d this value is found for almost all x_1 .

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Consider the linear fractional transformation:

$$T(z) = \frac{az + b}{cz + d} \quad (ad - bc = 1; c \neq 0)$$

which is to have the property of mapping the extended real axis $\mathbb{R}$ one-one onto itself. It is easily seen that a necessary and sufficient condition for this is that all of a, b, c, d be real. (Note that we must allow $\mathbb{R}$ to contain the ideal "point at infinity" which we take as $-\infty$. This will allow the point $-\frac{d}{c}$ to have an image. Thus $\mathbb{R} \equiv [-\infty, +\infty)$.)

This transformation will have one or two fixed points. In the former case if λ is the fixed point, it will be real and any sequence of real numbers satisfying $x_{n+1} = T(x_n)$ ($n = 1, 2, \dots$) will converge to λ . If there are two fixed points then either:

(a) they will both be real

or

(b) they will form a complex conjugate pair.

In case (a), any sequence generated as above will converge to one of the fixed points (call this one λ) whilst in case (b) there will not be convergence at all.

The behavior of the two convergent cases above can be expressed (more weakly) by asserting that

$$\frac{1}{N} \sum_1^N f(x_n) \rightarrow f(\lambda) \text{ as } N \rightarrow \infty$$

for any $f \in C(-\infty, \infty)$.

Our purpose here is to enquire about the existence and value of this limit in the non-convergent case (b) above. We shall show that, except for certain exceptional values of a, b, c, d , and for almost all x_1 , we shall have

$$\frac{1}{N} \sum_1^N f(x_n) \rightarrow \int_{-\infty}^{+\infty} \frac{Kf(x)}{cx^2 + (d-a)x - b} dx \quad (1)$$

whenever f is such that the right hand side exists as a Lebesgue integral. (The constant K is a normalizing constant whose value is such as to make the right hand side unity when $f(x) \equiv 1$.)

If α and $\bar{\alpha}$ are the two fixed points in case (b) then the transformation

$$y = T(x) \equiv \frac{ax + b}{cx + d}$$

can be written as

$$\frac{y - \alpha}{y - \bar{\alpha}} = e^{i\beta} \frac{x - \alpha}{x - \bar{\alpha}} \quad (2)$$

In this $0 < \beta < 2\pi$ since, c being non-zero, the identity transformation is excluded.

Define $S: \mathbb{R} \rightarrow [0, 2\pi)$ as follows:

$$S(x) = \arg\left(\frac{x - \alpha}{x - \bar{\alpha}}\right), \text{ with the principal value of } \arg \text{ lying in } [0, 2\pi).$$

Define $H_\beta: [0, 2\pi) \rightarrow [0, 2\pi)$ by $H_\beta(\phi) = \beta + \phi \pmod{2\pi}$

Clearly both S and H_β are one-one and onto. It is also well-known that, whenever β is not a rational multiple of π , H_β is ergodic with respect to ordinary Lebesgue probability measure m on $[0, 2\pi)$ and preserves this measure.

When x and y are real (2) is equivalent to

$$\arg\left(\frac{y - \alpha}{y - \bar{\alpha}}\right) = \beta + \arg\left(\frac{x - \alpha}{x - \bar{\alpha}}\right) \pmod{2\pi}$$

That is,

$$S(y) = H_\beta S(x)$$

or

$$y = S^{-1} H_\beta S(x)$$

So, since $y = T(x) \equiv \frac{ax + b}{cx + d}$, we see now that $T \equiv S^{-1} H_\beta S$.

To deal with this situation we shall next prove a lemma (stated in somewhat more general terms than is required for the present application).

LEMMA. Let $M_1(X_1, \mathcal{A}, m_1)$ and $M_2(X_2, \mathcal{B}, m_2)$ be two measure spaces with probability measures m_1 and m_2 . Let $S: X_1 \rightarrow X_2$ and $H: X_2 \rightarrow X_2$, both of these transformations being one-one and onto. Suppose that $E \subset X_1$ is m_1 -measurable if and only if $S(E) \subset X_2$ is m_2 -measurable and that $m_1(E) = m_2(S(E))$. Finally, suppose that H is m_2 -measure preserving and ergodic with respect to this measure. Then the transformation $T = S^{-1} H S$ is m_1 -measure preserving and ergodic with respect to m_1 .

PROOF. Since all our transformations are invertible, we may deal with the transformations themselves rather than their inverses.

(a) Let $E \subset X_1$ be m_1 -measurable.

Then $m_1(E) = m_2(S(E)) = m_2(HS(E)) = m_1(S^{-1}HS(E)) = m_1(T(E))$

This shows that T is m_1 -measure preserving.

(b) Let A be a subset of X_1 which is invariant under T and whose m_1 measure satisfies $0 < m_1(A) < 1$. Let $B = S(A)$. Then B is an invariant set under H because $H(B) = HS(A) = ST(A) = S(A) = B$. Also $m_1(A) = m_2(S(A)) = m_2(B)$ so that $0 < m_2(B) < 1$. But, since H is ergodic, this is impossible. Hence there cannot exist any set A satisfying both $T(A) = A$ and $0 < m_1(A) < 1$. This shows that T is ergodic with respect to the m_1 measure and the proof of the lemma is complete.

We now return to the particular application in hand. As we have already remarked, when β is not a rational multiple of π the transformation H_β of $[0, 2\pi)$ onto itself is ergodic with respect to Lebesgue probability measure m and preserves this measure. To apply the Lemma we let $X_1 \equiv \mathbb{R}$, $X_2 \equiv [0, 2\pi)$, $m_2 \equiv m$ (Lebesgue probability measure), $H \equiv H_\beta$ and $S \equiv$ the S of the application. It remains to find m_1 so that $m_1(E) = m(S(E))$ whenever $E \subset \mathbb{R}$.

Write $\phi \equiv S(x) = \arg\left(\frac{x-\alpha}{x-\bar{\alpha}}\right)$ so that $e^{i\phi} = \frac{x-\alpha}{x-\bar{\alpha}}$
By differentiation we find that

$$d\phi = \frac{\alpha - \bar{\alpha}}{i} \frac{1}{(x - \alpha)(x - \bar{\alpha})} dx$$

so that if $E \subset \mathbb{R} \equiv [-\infty, \infty)$ then

$$m(S(E)) = \frac{1}{2\pi} \int S(E) \cap [0, 2\pi) d\phi = \frac{\alpha - \bar{\alpha}}{2\pi i} \int E \cap [-\infty, \infty) \frac{1}{(x - \alpha)(x - \bar{\alpha})} dx$$

Since α and $\bar{\alpha}$ are the zeros of $cx^2 + (d-a)x - b$ this latter integral can be re-written in an obvious way and so we find that the required m_1 -measure is given by

$$m_1(E) = \int E \cap [-\infty, \infty) \frac{K}{cx^2 + (d-a)x - b} dx$$

with K chosen to make this a probability measure.

Applying the Lemma we see that the transformation T preserves this measure and is ergodic with respect to it.

Finally, knowing this, we can apply the Individual Ergodic Theorem [1] to conclude the following:

THEOREM. Let $T(z) = \frac{az+b}{cz+d}$ ($ad-bc=1: c \neq 0$) map the extended real axis $\mathbb{R}$ onto itself and let it have a pair of complex conjugate fixed points. If a, b, c, d are such as to make β in (2) not a rational multiple of π then, for almost all x_1 (in the m_1 sense *), the sequence defined by

$$x_{n+1} = \frac{ax_n + b}{cx_n + d} \quad (n = 1, 2, \dots) \quad (ad-bc=1, c \neq 0)$$

will have the property stated in (1) above.

NOTE: *By referring to the definition of m_1 measure above, "almost all in the m_1 sense" can be seen to be equivalent to "almost all in the sense of Lebesgue measure on $\mathbb{R}$."

REFERENCES

1. PARRY, W., Topics in Ergodic Theory, Cambridge Univ. Press, 1981.