ABSTRACT. Let $X$ be an abstract set and $\mathcal{L}$ a lattice of subsets of $X$. To each lattice-regular measure $\mu$, we associate two induced measures $\hat{\mu}$ and $\check{\mu}$ on suitable lattices of the Wallman space $I_{\mathcal{R}}(\mathcal{L})$ and another measure $\mu'$ on the space $I_{\mathcal{P}}(\mathcal{L})$. We will investigate the reflection of smoothness properties of $\mu$ onto $\hat{\mu}$, $\check{\mu}$ and $\mu'$; and try to set some new criterion for repleteness and measure repleteness.

KEY WORDS AND PHRASES. Replete and measure replete lattices, Lattice regular measure, Wallman space and remainder, $\sigma$-smooth, $\tau$-smooth and tight measures, purely finitely additive measures, purely $\sigma$-additive measures, purely $\tau$-additive measures.

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1. INTRODUCTION: Let $X$ be an abstract set and $\mathcal{L}$ a lattice of subsets of $X$. To each lattice regular measure $\mu$, we associate following Bachman and Szeto [1], two induced measures $\hat{\mu}$ and $\check{\mu}$ on suitable lattices of the Wallman space $I_{\mathcal{R}}(\mathcal{L})$ of $(X, \mathcal{L})$; we also associate to $\mu$ a measure $\mu'$ on the space $I_{\mathcal{P}}(\mathcal{L})$(see below for definitions). We give in section 2, a brief review of the lattice notation and terminology relevant to the paper. We will be consistent with the standard terminology as used, for example, in Alexandroff [2], Frolik [3], Grassi [4], and Nöbeling [5]. We also give a brief review of the principal Theorems of [1] that we need in order to make the paper reasonably self-contained.

2. DEFINITIONS AND NOTATIONS

Let $X$ be an abstract set, then $\mathcal{L}$ is a lattice of subsets of $X$ if for $A, B \subseteq X$ then $A \cup B \in \mathcal{L}$ and $A \cap B \in \mathcal{L}$. Throughout this work we will always assume that $\emptyset$ and $X$ are in $\mathcal{L}$. If $A \subseteq X$ then we will denote the complement of $A$ by $A'$ i.e. $A' = X - A$. If $\mathcal{L}$ is a Lattice of subsets of $X$ then $\mathcal{L}'$ is defined $\mathcal{L}' = \{L' \mid L \in \mathcal{L}\}$.

Lattice Terminology

DEFINITIONS 2.1. Let $\mathcal{L}$ be a Lattice of subsets of $X$. We say that:

1. $\mathcal{L}$ is a $\delta$-Lattice if it is closed under countable intersections.
2. $\mathcal{L}$ is separating or $T_1$ if for $x, y \in X$; $x \neq y$ then $\exists L \in \mathcal{L}$ such that $x \in L$ and $y \notin L$.
3. $\mathcal{L}$ is Hausdorff or $T_2$ if for $x, y \in X$; $x \neq y$ then $\exists A, B \in \mathcal{L}$ such that $x \in A'$, $y \in B'$ and $A' \cap B' = \emptyset$.
4. $\mathcal{L}$ is disjunctive if for $x \in X$ and $L \in \mathcal{L}$ where $x \notin L$; $\exists A, B \in \mathcal{L}$ such that $x \in A, L \subseteq B$ and $A \cap B = \emptyset$.
5. $\mathcal{L}$ is regular if for $x \in X, L \in \mathcal{L}$ and $x \notin L$; $\exists A, B \in \mathcal{L}$ such that $x \in A', L \subseteq B'$ and $A' \cap B' = \emptyset$. 
6- \( \mathcal{L} \) is normal if for \( A, B \in \mathcal{L} \), where \( A \cap B = \emptyset \), \( \bar{A}, \bar{B} \in \mathcal{L} \) such that \( A \subset \bar{A}', B \subset \bar{B}' \) and \( \bar{A}' \cap \bar{B}' = \emptyset \).

7- \( \mathcal{L} \) is compact if \( X = \bigcup_{\alpha} L'_{\alpha} \) where \( L_{\alpha} \in \mathcal{L} \) then there exists a finite number of \( L_{\alpha} \) that cover \( X \) i.e.

\[
X = \bigcup_{\alpha} L'_{\alpha} \text{ where } L'_{\alpha} \in \mathcal{L}.
\]

\( A(\mathcal{L}) \) = the algebra generated by \( \mathcal{L} \).

\( \sigma(\mathcal{L}) \) = the \( \sigma \)-algebra generated by \( \mathcal{L} \).

\( \delta(\mathcal{L}) \) = the Lattice of countable intersections of sets of \( \mathcal{L} \).

\( \tau(\mathcal{L}) \) = the Lattice of arbitrary intersection of sets of \( \mathcal{L} \).

\( \rho(\mathcal{L}) \) = the smallest class containing \( \mathcal{L} \) and closed under countable unions and intersections.

If \( A \in \mathcal{A}(\mathcal{L}) \) then \( A = \bigcup_{i=1}^{\infty} (L_i - L'_i) \) where the union is disjoint and \( L_i, L'_i \in \mathcal{L} \).

**Measure Terminology**

Let \( \mathcal{L} \) be a lattice of subsets of \( X \). \( M(\mathcal{L}) \) will denote the set of finite valued bounded finitely additive measures on \( \mathcal{A}(\mathcal{L}) \). Clearly since any measure in \( M(\mathcal{L}) \) can be written as a difference of two non-negative measures there is no loss of generality in assuming that the measures are non-negative, and we will assume so throughout this paper.

**DEFINITIONS 2.2.**

1- A measure \( \mu \in M(\mathcal{L}) \) is said to be \( \sigma \)-smooth on \( \mathcal{L} \) if for \( L_n \in \mathcal{L} \) and \( L_n \downarrow \emptyset \) then \( \mu(L_n) \rightarrow 0 \).

2- A measure \( \mu \in M(\mathcal{L}) \) is said to be \( \sigma \)-smooth on \( \mathcal{A}(\mathcal{L}) \) if for \( A_n \in \mathcal{A}(\mathcal{L}), A_n \downarrow \emptyset \) then \( \mu(A_n) \rightarrow 0 \).

3- A measure \( \mu \in M(\mathcal{L}) \) is said to be \( \tau \)-smooth on \( \mathcal{L} \) if for \( L_n \in \mathcal{L}, \alpha \in A, L_\alpha \downarrow \emptyset \) then \( \mu(L_\alpha) \rightarrow 0 \).

4- A measure \( \mu \in M(\mathcal{L}) \) is said to be \( \mathcal{L} \)-regular if for any \( A \in \mathcal{A}(\mathcal{L}) \)

\[
\mu(A) = \sup_{L, C \in \mathcal{L}} \mu(L)
\]

If \( \mathcal{L} \) is a lattice of subsets of \( X \), then we will denote by:

- \( M_\mathcal{R}(\mathcal{L}) \) = the set of \( \mathcal{L} \)-regular measures of \( M(\mathcal{L}) \)
- \( M_\sigma(\mathcal{L}) \) = the set of \( \sigma \)-smooth measures on \( \mathcal{L} \) of \( M(\mathcal{L}) \)
- \( M^\sigma(\mathcal{L}) \) = the set of \( \sigma \)-smooth measures on \( \mathcal{A}(\mathcal{L}) \) of \( M(\mathcal{L}) \)
- \( M_\mathcal{R}^\sigma(\mathcal{L}) \) = the set of regular measures of \( M^\sigma(\mathcal{L}) \)
- \( M_\tau(\mathcal{L}) \) = the set of \( \tau \)-smooth measures on \( \mathcal{L} \) of \( M_\mathcal{R}(\mathcal{L}) \)
- \( M_\tau^\sigma(\mathcal{L}) \) = the set of tight measures on \( \mathcal{L} \) of \( M_\mathcal{R}^\sigma(\mathcal{L}) \).

Clearly

\[
M_\mathcal{R}^\tau(\mathcal{L}) \subset M_\sigma^\tau(\mathcal{L}) \subset M_\sigma(\mathcal{L})
\]

**DEFINITION 2.3.** If \( A \in \mathcal{A}(\mathcal{L}) \) then \( \mu_x \) is the measure concentrated at \( x \in X \).

\[
\mu_x(A) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}
\]

\( I(\mathcal{L}) \) is the subset of \( M(\mathcal{L}) \) which consists of non-trivial zero-one measures which are finitely additive on \( \mathcal{A}(\mathcal{L}) \).

- \( I_\mathcal{R}(\mathcal{L}) \) = the set of \( \mathcal{L} \)-regular measures of \( I(\mathcal{L}) \)
- \( I_\sigma(\mathcal{L}) \) = the set of \( \sigma \)-smooth measures on \( \mathcal{L} \) of \( I(\mathcal{L}) \)
- \( I^\sigma(\mathcal{L}) \) = the set of \( \sigma \)-smooth measures on \( \mathcal{A}(\mathcal{L}) \) of \( I(\mathcal{L}) \)
\( I_s(L) \) - the set of \( \tau \)-smooth measures on \( L \) of \( I(L) \)
\( I_r(L) \) - the set of \( L \)-regular measures of \( I(L) \)
\( I^r(L) \) - the set of \( L \)-regular measures of \( I(L) \)

**Definition 2.4:** If \( \mu \in M(L) \) then we define the support of \( \mu \) to be:
\[
S(\mu) = \cap \{ L \in L | \mu(L) = \mu(X) \}.
\]
Consequently if \( \mu \in I(L) \)
\[
S(\mu) = \cap \{ L \in L | \mu(L) = 1 \}
\]

**Definition 2.5.** If \( L \) is a lattice of subsets of \( X \), we say that \( L \) is replete if for any \( \mu \in I_r^r(L) \) then
\[
S(\mu) \neq \emptyset.
\]

**Definition 2.6.** Let \( L \) be a lattice of subsets of \( X \). We say that \( L \) is measure replete if \( S(\mu) \neq \emptyset \) for all \( \mu \in M_r^r(L) \), \( \mu \neq 0 \).

**Separation Terminology**

Let \( L_1 \) and \( L_2 \) be two lattices of subsets of \( X \).

**Definition 2.7.** \( L_1 \) separates \( L_2 \) if for \( A_2, B_2 \in L_2 \) and \( A_2 \cap B_2 = \emptyset \) then there exists \( A_1, B_1 \in L_1 \) such that \( A_2 \subseteq A_1, B_2 \subseteq B_1 \) and \( A_1 \cap B_2 = \emptyset \).

**Remark 2.1.** We now list few known facts found in [1] which will enable us to characterize some previously defined properties in a measure theoretic fashion.
1. \( L \) is disjunctive if and only if \( \mu_x \in I_r(L), \forall x \in X \).
2. \( L \) is regular if and only if for any \( \mu_1, \mu_2 \in I(L) \) such that \( \mu_1 \sim \mu_2 \) on \( L \) we have \( S(\mu_1) = S(\mu_2) \).
3. \( L \) is \( T_2 \) if and only if \( S(\mu) = \emptyset \) or a singleton for any \( \mu \in I(L) \).
4. \( L \) is compact if and only if \( S(\mu) \neq \emptyset \) for any \( \mu \in I_r(L) \).

**3. The Induced Measures**

If \( L \) is a disjunctive lattice of subsets of an abstract set \( X \) then there is a Wallman space associated with it. We will briefly review the fundamental properties of this Wallman space, and then associate with a regular lattice measure \( \mu \), two measures \( \hat{\mu} \) and \( \check{\mu} \) on certain algebras in the Wallman space (see [1]). We then investigate how properties of \( \mu \) reflect to those of \( \hat{\mu} \) and \( \check{\mu} \), and conversely, then give a variety of applications of these results. Let \( X \) be an abstract set and \( L \) a disjunctive lattice of subsets of \( X \) such that \( \emptyset \) and \( X \) are in \( L \). For any \( A \in \mathcal{A}(L) \), define \( W(A) = \{ \mu \in I_r(L) | \mu(A) = 1 \} \). If \( A, B \in \mathcal{A}(L) \) then

1. \( W(A \cup B) = W(A) \cup W(B) \).
2. \( W(A \cap B) = W(A) \cap W(B) \).
3. \( W(A^c) = W(A)^c \).
4. \( W(A) \subseteq W(B) \) if and only if \( A \subseteq B \).
5. \( W(A) = W(B) \) if and only if \( A = B \).
6. \( W(\mathcal{A}(L)) = \mathcal{A}(W(L)) \).

Let \( W(L) = \{ W(L), L \in L \} \). Then \( W(L) \) is a compact lattice of \( I_r(L) \), and \( I_r(L) \) with \( \tau W(L) \) as the topology of closed sets is a compact \( T_1 \) space (the Wallman space) associated with the pair \( X, L \). It is a \( T_2 \)-space if and only if \( L \) is normal. For \( \mu \in M(L) \) we define \( \hat{\mu} \) on \( \mathcal{A}(W(L)) \) by:
\[
\hat{\mu}(W(A)) = \mu(A) \quad \text{where} \quad A \in \mathcal{A}(L).
\]
Then \( \hat{\mu} \in M(W(L)) \), and \( \mu \in M_r(W(L)) \) if and only if \( \mu \in M_r(L) \).
Finally, since \( \tau W(\mathcal{L}) \) and \( W(\mathcal{L}) \) are compact lattices, and \( W(\mathcal{L}) \) separates \( \tau W(\mathcal{L}) \), then \( \hat{\mu} \) has a unique extension to \( \hat{\mu} \in M_\mu(\tau W(\mathcal{L})) \). We note that by compactness \( \hat{\mu} \) and \( \hat{\mu} \) are in \( M_\mu(W(\mathcal{L})) \) and \( M_\mu(\tau W(\mathcal{L})) \) respectively, where they are certainly \( \tau \)-smooth and of course \( \sigma \)-smooth. \( \hat{\mu} \) can be extended to \( \sigma(W(\mathcal{L})) \) where it is \( \delta W(\mathcal{L}) \)-regular; while \( \hat{\mu} \) can be extended to \( \sigma(\tau W(\mathcal{L})) \), the Borel sets of \( I(\mathcal{L}) \), and is \( \tau W(\mathcal{L}) \)-regular on it. One is now concerned with how further properties of \( \mu \) reflect over to \( \hat{\mu} \) and \( \hat{\mu} \) respectively. The following are known to be true (see [1]) and we list them for the reader’s convenience.

**Theorem 3.1.** Let \( \mathcal{L} \) be a separating and disjunctive lattice of subsets of \( X \), and let \( \mu \in M_\mu(\mathcal{L}) \). Then

1. \( \mu \in M_\mu^*(\mathcal{L}) \) if and only if \( \hat{\mu}^*(X) = \hat{\mu}(I(\mathcal{L})) \).
2. \( \mu \in M_\mu^*(\mathcal{L}) \) if and only if \( \hat{\mu}^*(X) = \hat{\mu}(I(\mathcal{L})) \).

3. If \( \mathcal{L} \) is also normal (or \( T_\sigma \)) then \( \mu \in M_\mu^*(\mathcal{L}) \) if and only if \( X \) is \( \mu^* \)-measurable and \( \hat{\mu}^*(X) = \hat{\mu}(I(\mathcal{L})) \).

We now give some further results related to the induced measures \( \hat{\mu} \) and \( \hat{\mu} \).

**Theorem 3.2.** Let \( \mathcal{L} \) be a separating and disjunctive lattice, and \( \mu \in M_\mu(\mathcal{L}) \) then \( \hat{\mu} \) is \( \tau W(\mathcal{L}) \)-regular on \( (\tau W(\mathcal{L}))' \).

**Proof.** We know that \( W(\mathcal{L}) \) and \( \tau W(\mathcal{L}) \) are compact lattices and that \( W(\mathcal{L}) \) separates \( \tau W(\mathcal{L}) \). Since \( \mu \in M_\mu(\mathcal{L}) \) then \( \hat{\mu} \in M_\mu(W(\mathcal{L})) \). Extend \( \hat{\mu} \) to \( \tau W(\mathcal{L}) \). The extension is

\[ \hat{\mu} \in M_\mu(\tau W(\mathcal{L}))-M_\mu(W(\mathcal{L}))-M_\mu^*(\tau W(\mathcal{L}))-M_\mu^*(W(\mathcal{L})). \]

Let \( 0 \in (\tau W(\mathcal{L}))' \) then since \( \hat{\mu} \in M_\mu(\tau W(\mathcal{L})) \) there exists \( F \in \tau W(\mathcal{L}), F \subset 0 \) and

\[ |\hat{\mu}(0) - \hat{\mu}(F)| < \varepsilon; \varepsilon > 0. \]

Since \( F \in \tau W(\mathcal{L}), F = \bigcap_{\alpha} W(L_\alpha), L_\alpha \in \mathcal{L} \). Also since \( F \subset 0 \) then \( F \cap 0' = \emptyset \) i.e. \( \bigcap_{\alpha} W(L_\alpha) \cap 0' = \emptyset \) by compactness there must exist \( \alpha_0 \in \Lambda \) such that \( W(L_{\alpha_0}) \cap 0' = \emptyset \) thus \( F \subset W(L_{\alpha_0}) \subset 0' = 0 \) so

\[ |\hat{\mu}(0) - \hat{\mu}(W(L_{\alpha_0}))| = \varepsilon \]

i.e. \( \hat{\mu} \) is \( W(\mathcal{L}) \)-regular on \( (\tau W(\mathcal{L}))' \).

**Theorem 3.3.** Let \( \mu \in M_\mu(\mathcal{L}) \) then \( \hat{\mu} \) is \( \tau W(\mathcal{L}) \)-regular.

**Proof.** Since \( \mu \in M_\mu(\mathcal{L}) \) and \( W(\mathcal{L}) \) is compact then \( \hat{\mu} \in M_\mu(W(\mathcal{L}))-M_\mu^*(W(\mathcal{L})) \) and since \( W(\mathcal{L}) \) separates \( \tau W(\mathcal{L}) \) and \( \tau W(\mathcal{L}) \) is compact then \( \hat{\mu} \in M_\mu(\tau W(\mathcal{L}))-M_\mu^*(\tau W(\mathcal{L})) \) furthermore \( \hat{\mu} \) extends \( \hat{\mu} \) to \( \tau W(\mathcal{L}) \) uniquely. Let \( F \in \tau W(\mathcal{L}) \) then

\[ \hat{\mu}(F) = \inf \sum_{i} \hat{\mu}(A_i), F \subset \bigcup_{i} A_i \text{ and } A_i \in \mathcal{A}[W(\mathcal{L})] \]

and since \( \hat{\mu} \in M_\mu^*(W(\mathcal{L})) \) then

\[ \hat{\mu}(A_i) = \inf \hat{\mu}[W(L_i')], A_i \subset W(L_i'), L_i \in \mathcal{L} \]

thus \( F \subset \bigcup_{i} W(L_i') \) but since \( W(\mathcal{L}) \) is compact then \( F \subset \bigcup_{i} W(L_i') = W(L') \) where \( L \in \mathcal{L} \) and

\[ \hat{\mu}(F) = \inf \hat{\mu}[W(L')] \text{; } F \subset W(L') \text{ and } L \in \mathcal{L} \]

Now \( F \subset W(L') \Rightarrow F \cap W(L) = \emptyset \) then since \( W(\mathcal{L}) \) separates \( \tau W(\mathcal{L}) \) \( \exists L \in \mathcal{L} \) such that \( F \subset W(L) \) and \( W(L) \cap W(L) = \emptyset \). Therefore \( W(L) \subset W(L') \) and hence
\[ \tilde{\mu}^*(F) = \inf_{\alpha} \tilde{\mu}(W(L)) \] where \( F \subset W(L) \), \( \tilde{\mu}^* \) is regular on \( \tau W(L) \). On the other hand since \( \tau W(L) \) is \( \delta \) then

\[ F = \bigcap_{\alpha} W(L_\alpha) \text{ and } \tilde{\mu} \left( \bigcap_{\alpha} W(L_\alpha) \right) = \inf_{\alpha} \tilde{\mu}(W(L_\alpha)) = \inf_{\alpha} \tilde{\mu}(W(L_\alpha)) \]

where \( F \subset W(L_\alpha), \alpha \in \mathcal{L} \). Therefore \( \tilde{\mu}^* = \tilde{\mu} \) on \( \tau W(L) \).

**Theorem 3.4.** Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be two lattices of subsets of \( X \) such that \( \mathcal{L}_1 \subset \mathcal{L}_2 \) and \( \mathcal{L}_1 \) separates \( \mathcal{L}_2 \). If \( v \in \mathcal{M}^0(\mathcal{L}_2) \) then \( v = \mu^* \) on \( \mathcal{L}_2 \) and \( v = \mu_\alpha \) on \( \mathcal{L}_2 \) where \( \mu = \mu_{\mathcal{L}_1} \).

**Proof.** Let \( v \in \mathcal{M}^0(\mathcal{L}_2) \) then since \( \mathcal{L}_1 \) separates \( \mathcal{L}_2, \mu \in \mathcal{M}^0(\mathcal{L}_1) \). Since \( \mathcal{L}_1 \subset \mathcal{L}_2 \) then \( \alpha(\mathcal{L}_2) \subset \alpha(\mathcal{L}_1) \); Let \( \mathcal{L} \subset X \) then

\[ v^*(E) = \inf_{E \subset B, B \in \alpha(\mathcal{L}_1)} v(B) \leq \inf_{E \subset A, A \in \alpha(\mathcal{L}_1)} v(A) = \mu^*(E) \]

therefore, \( v^* \leq \mu^* \). Now on \( \mathcal{L}_2, v^* = \mu^* \). Suppose \( \exists \mathcal{L}_2 \subset \mathcal{L}_2 \) such that \( v(\mathcal{L}_2) < \mu(\mathcal{L}_2) \) then since \( v \in \mathcal{M}^0(\mathcal{L}_2) \), \( v(\mathcal{L}_2) = \inf v(\mathcal{L}_2, \mathcal{L}_2 \subset \mathcal{L}_2 \) and \( \mathcal{L}_2 \subset \mathcal{L}_2 \) then \( \mathcal{L}_2 \cap \mathcal{L}_2 = \emptyset \) and by separation \( \exists \mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{L}_1 \) such that \( \mathcal{L}_2 \subset \mathcal{L}_1, \mathcal{L}_1 \subset \mathcal{L}_2 \) and therefore

\[ v(\mathcal{L}_2) = \inf_{\alpha} \mu(\mathcal{L}_1, \mathcal{L}_2) \text{ where } \mathcal{L}_2 \subset \mathcal{L}_1 \]

\[ = \inf_{\alpha} v(\mathcal{L}_2) \text{ where } \mathcal{L}_2 \subset \mathcal{L}_1 \]

\[ < \mu^*(\mathcal{L}_2) \]

\[ \forall \epsilon > 0 \exists \mathcal{L}_1 \subset \mathcal{L}_1 \text{ such that } \mathcal{L}_2 \subset \mathcal{L}_1 \text{ and } \mu(\mathcal{L}_1) - \epsilon < v(\mathcal{L}_2) < \mu(\mathcal{L}_1) \] but since \( \mathcal{L}_2 \subset \mathcal{L}_1 \) then \( \mu^*(\mathcal{L}_2) \leq \mu(\mathcal{L}_1) < v(\mathcal{L}_2) + \epsilon \) which is a contradiction to our assumption. Therefore \( v = \mu^* \) on \( \mathcal{L}_2 \) and thus \( v = \mu_\alpha \) on \( \mathcal{L}_2 \). This theorem is a generalization of the previous one in which we used the compactness of \( W(\mathcal{L}) \) to have a regular restriction of the measure. Next consider the space \( I^0(\mathcal{L}) \) and the induced measure \( \mu' \).

**Definition 3.1.** Let \( \mathcal{L} \) be a disjunctive lattice of subsets of \( X \).

1) \( W_0(L) = \{ \mu \in I^0_\mathcal{R}(\mathcal{L}) \mid \mu(L) = 1 \} : L \subset \mathcal{L} \)
2) \( W_0(\mathcal{L}) = \{ W_0(L), L \subset \mathcal{L} \} \)
3) \( W_0(A) = \{ \mu \in I^0_\mathcal{R}(\mathcal{L}) \mid \mu(A) = 1 \}, A \in \mathcal{A}(\mathcal{L}) \)
4) \( W_0(\mathcal{L}) = W(\mathcal{L}) \cap I^0_\mathcal{R}(\mathcal{L}) \)

The following properties hold:

**Proposition 3.1.** Let \( \mathcal{L} \) be a disjunctive lattice then for \( A, B \in \mathcal{A}(\mathcal{L}) \)
1) \( W_0(A \cap B) = W_0(A) \cup W_0(B) \)
2) \( W_0(A \cup B) = W_0(A) \cap W_0(B) \)
3) \( W_0(A') = W_0(A)' \)
4) \( W_0(A) \subset W_0(B) \) if and only if \( A \subset B \)
5) \( \mathcal{A}(W_0(\mathcal{L})) = W_0[\mathcal{A}(\mathcal{L})] \)

The proof is the same as for \( W(\mathcal{L}) \) by simply using the properties of \( W(\mathcal{L}) \) and the fact that \( W_0(A) = W(A) \cap I^0_\mathcal{R}(\mathcal{L}) \) and \( W_0(B) = W(B) \cap I^0_\mathcal{R}(\mathcal{L}) \).

**Remark 3.1.** It is not difficult to show that \( \mathcal{A}(W_0(\mathcal{L})) = W_0[\mathcal{A}(\mathcal{L})] \). Also, for each \( \mu \in M(\mathcal{L}) \) we define \( \mu' \) on \( \mathcal{A}(W_0(\mathcal{L})) \) as follows:
\( \mu'[W_o(A)] = \mu(A) \) where \( A \in \mathcal{A} \).

\( \mu' \) is defined and the map \( \mu \to \mu' \) from \( M(\mathcal{L}) \) to \( M(W_o(\mathcal{L})) \) is onto. In addition, it can readily be checked that,

**Theorem 3.5.** Let \( \mathcal{L} \) be disjunctive then

1) \( \mu \in M(\mathcal{L}) \) if and only if \( \mu' \in M(W_o(\mathcal{L})) \)

2) \( \mu \in M_\delta(\mathcal{L}) \) if and only if \( \mu' \in M_\delta(W_o(\mathcal{L})) \)

3) \( \mu \in M_\alpha(\mathcal{L}) \) if and only if \( \mu' \in M_\alpha(W_o(\mathcal{L})) \)

4) \( \mu \in M^\ast(\mathcal{L}) \) if and only if \( \mu' \in M^\ast(W_o(\mathcal{L})) \)

5) \( \mu \in M_\theta(\mathcal{L}) \) if and only if \( \mu' \in M_\theta(W_o(\mathcal{L})) \)

**Theorem 3.6.** Let \( \mathcal{L} \) be a separating and disjunctive lattice of subsets of \( X \), and let \( \mu \in M_\beta^\ast(\mathcal{L}) \).

Then

1. \( \mu' \in M_\beta^\ast(W_o(\mathcal{L})) \) if and only if \( \check{\mu}(I_\beta^\ast(\mathcal{L})) = \check{\mu}(I_\beta(\mathcal{L})) \).

2. If \( \mathcal{L} \) is also normal or \( T_2 \) then \( \mu' \in M_\beta^\ast(W_o(\mathcal{L})) \) if and only if \( I_\beta(\mathcal{L}) \) is \( \check{\mu}^\ast \)-measurable and \( \check{\mu}^\ast(I_\beta^\ast(\mathcal{L})) = \check{\mu}(I_\beta(\mathcal{L})) \).

We note some consequences.

**Corollary 3.1.** If \( \mathcal{L} \) is a separating, disjunctive and replete lattice of subsets of \( X \), then \( \mu' \in [M_\beta^\ast(\mathcal{L})] \) implies \( \mu \in M_\beta^\ast(\mathcal{L}) \).

**Proof.** Since \( \mathcal{L} \) is replete then \( X = I_\beta^\ast(\mathcal{L}) \) then from the previous theorem we have

\[ \check{\mu}(I_\beta(\mathcal{L})) = \check{\mu}^\ast(I_\beta^\ast(\mathcal{L})) = \check{\mu}(X) \]

i.e. \( \mu \in M_\beta^\ast(\mathcal{L}) \) from theorem 3.1.

**Corollary 3.2.** Let \( \mathcal{L} \) be separating and disjunctive. If \( \mu' \in M_\beta^\ast(W_o(\mathcal{L})) \Rightarrow \mu \in M_\beta^\ast(\mathcal{L}) \) then \( \mathcal{L} \) is replete.

**Proof.** Let \( \mu \in I_\beta^\ast(\mathcal{L}) \) then since \( W_o(\mathcal{L}) \) is replete \( \mu' \in I_\beta^\ast(W_o(\mathcal{L})) \) then by hypothesis \( \mu \in I_\beta^\ast(\mathcal{L}) \).

Therefore \( I_\beta^\ast(\mathcal{L}) = I_\beta^\ast(W_o(\mathcal{L})) \) or \( \mathcal{L} \) is replete. If we combine the two corollaries we get the following:

**Theorem 3.7.** Let \( \mathcal{L} \) be separating and disjunctive. Then \( \mathcal{L} \) is replete if and only if \( \mu' \in M_\beta^\ast(W_o(\mathcal{L})) \Rightarrow \mu \in M_\beta^\ast(\mathcal{L}) \).

**Theorem 3.8.** Let \( \mathcal{L} \) be a separating, disjunctive, normal and replete lattice. Then

\( \mu' \in M_\beta^\ast[W_o(\mathcal{L})] \) if and only if \( \mu \in M_\beta^\ast(\mathcal{L}) \).

**Proof.**

1. Let \( \mu' \in M_\beta^\ast(W_o(\mathcal{L})) \) then since \( \mathcal{L} \) is replete then \( X = I_\beta^\ast(\mathcal{L}) \) and \( X \) is \( \check{\mu}^\ast \)-measurable and

\[ \check{\mu}^\ast(I_\beta^\ast(\mathcal{L})) = \check{\mu}(I_\beta(\mathcal{L})) = \check{\mu}(X) \]

then by theorem 3.1 we get that \( \mu \in M_\beta^\ast(\mathcal{L}) \).

2. Conversely suppose \( \mu \in M_\beta^\ast(\mathcal{L}) \) then from theorem 3.1 we get that

\[ \check{\mu}(X) = \check{\mu}(I_\beta(\mathcal{L})) \]
SPECIAL MEASURES AND REPLETENESS

and \( X \) is \( \tilde{\mu}^* \)-measurable but \( X \subset I_{\mu}^*(L) \subset I_{\mu}(L) \) therefore \( \tilde{\mu}^*(I_{\mu}^*(L)) = \tilde{\mu}(I_{\mu}(L)) \), then since \( L \) is replete \( X = I_{\mu}^*(L) \) so \( \tilde{\mu}^*(X) = \tilde{\mu}^*(I_{\mu}^*(L)) = \tilde{\mu}(I_{\mu}(L)) \) then from theorems 3.1 and 3.7 \( \mu^* \in M_\mu^*[W,(L)] \).

4. SPECIAL MEASURES AND REPLETENESS

In this section we define a purely finitely additive measure (p. f. a.), a purely \( \sigma \)-additive measure (p. \( \sigma \). a.) and a purely \( \tau \)-additive measure (p. \( \tau \). a.) and for each type we give a characterization theorem. Then we will define strong \( \sigma \)-additive measures (s. \( \sigma \). a.) and (s. \( \tau \). a.) measures and give for each a characterization theorem. Finally we will investigate relationships among these measures under repleteness.

**LEMMA 4.1.** Let \( L \) be a lattice of subsets of \( X \) and \( \mu \in M_\mu(L) \).

1. Consider \( \mu \) on \( \delta[W,(L)] \); we saw in earlier work that \( \mu \) is \( \delta(W,L) \) regular on \( \delta[W,(L)] \).

Let \( H \subset I_{\mu}(L) \) such that \( \mu^*(H) = \alpha \neq 0 \) then \( \exists \mu \) countably additive on \( \delta[W,(L)] \) and \( \tau - W(L) \) regular such that \( 0 < \mu \leq \mu \) and \( \mu^*(H) = \mu(I_{\mu}(L)) = \alpha \neq 0 \).

2. Consider \( \mu \) in \( \delta[W,(L)] \); we say that \( \mu \) is \( \tau W(L) \) regular on \( \delta[W,(L)] \).

Let \( H \subset I_{\mu}(L) \) such that \( \mu^*(H) = \alpha \neq 0 \) then \( \exists \mu \) countably additive on \( \tau W(L) \) regular on \( \delta[W,(L)] \) such that \( 0 < \mu \leq \mu \) and \( \mu^*(H) = \mu(I_{\mu}(L)) = \alpha \).

**DEFINITION 4.1.**

1. Let \( \mu \in M_\mu(L) \); we say that \( \mu \) is p. f. a. if for \( \gamma \in M_\mu(L) \) and \( 0 \leq \gamma \leq \mu \) on \( A(L) \) then \( \gamma = 0 \).

2. Let \( \mu \in M_\mu(L) \); we say that \( \mu \) is p. \( \sigma \). a. if for \( \gamma \in M_\mu(L), \gamma \tau \)-smooth on \( L \) and \( 0 \leq \gamma \leq \mu \) then \( \gamma = 0 \).

**THEOREM 4.1.** Let \( L \) be a separating and disjunctive lattice and \( \mu \in M_\mu(L) \) then:

1. \( \mu \) is p. f. a. \( \Rightarrow \mu^*(X) = 0 \).

2. \( \mu \) is p. \( \sigma \). a. \( \Rightarrow \mu^*(X) = 0 \).

If we further assume that \( L \) is \( \delta \) and \( \sigma(L) = \rho(L) \) then the converses are true.

**PROOF.** The proof will be given only for part (1) and is similar for the second one.

1. Suppose \( \mu \) is purely finitely additive. If \( \mu^*(X) = \alpha \neq 0 \) then from previous Lemma 4.1 there exists \( \rho \in M_\rho[W,L)] = M_\rho^*[W,L)] \) such that

\[
0 \leq \rho \leq \mu \text{ and } \rho^*(X) = \rho(I_{\rho}(L)) = \alpha;
\]

then

\[
\rho - \gamma \quad \text{and} \quad \gamma \in M_\rho^*[L) \text{ so}
\]

\[
0 \leq \rho - \gamma \leq \mu \Rightarrow 0 \leq \gamma \leq \mu \Rightarrow \gamma = 0
\]

from the definition of purely finitely additive which is a contradiction because

\[
\hat{\gamma}[I_{\rho}(L)] = \alpha \neq 0 \text{ and therefore } \hat{\mu}^*(X) = 0.
\]

2. Conversely if \( \hat{\mu}^*(X) = 0 \) and \( 0 \leq \gamma \leq \mu \) on \( A(L) \) where \( \gamma \in M_\gamma(L) \) and \( L \) is \( \delta \) and \( \rho(L) = \sigma(L) \) then \( \gamma \in M_\gamma^*(L) \) and \( 0 \leq \gamma \leq \mu \) on \( A(W,L) \) then \( 0 \leq \gamma \leq \mu \) on \( A(L) \); and therefore

\[
0 \leq \gamma \leq \mu^* \text{ and since } \hat{\mu}^*(X) = 0 \Leftrightarrow \hat{\gamma}^*(X) = 0 = \hat{\gamma}[I_{\rho}(L)]
\]

hence \( \gamma = 0 \) i.e. \( \mu \) is purely finitely additive.

**DEFINITIONS 4.2.** Let \( L \) be any lattice of subsets of \( X \).
1. Let $\mu \in M_p(\mathcal{L})$, we say that $\mu$ is s. f. a. if for $\gamma$ such that $0 \leq \gamma \leq \mu$ on $\mathcal{A}(\mathcal{L})$ and $\gamma' \in M'[W_0(\mathcal{L})]$ then $\gamma = 0$.

2. Let $\mu \in M_{p}^\ast(\mathcal{L})$, we say that $\mu$ is s. o. a. if for $\gamma$ such that $0 \leq \gamma \leq \mu$ on $\mathcal{A}(\mathcal{L})$ and $\gamma' \in M'[W_0(\mathcal{L})]$, $\gamma'$ t-smooth on $W_0(\mathcal{L})$ then $\gamma = 0$.

**Lemma 4.2.** Let $\mathcal{L}$ be a disjunctive lattice of subsets of $X$. If $\lambda \in M_p(\tau W(\mathcal{L})) = M_p^\ast(\tau W(\mathcal{L}))$ and $\lambda^\ast(I_p(\mathcal{L})) = \lambda(I_p(\mathcal{L}))$ then $\exists \mu \in M(p(\mathcal{L}))$ such that $\lambda = \mu$ and $\mu' \in M_{p}^\ast[W_0(\mathcal{L})]$. The proof is not difficult.

**Theorem 4.2.** Let $\mathcal{L}$ be a disjunctive lattice of subsets of $X$. Let $\mu \in M_{p}^\ast(\mathcal{L})$ then:

1. If $\mu$ is s. o. a. then $\mu^\ast(I_p(\mathcal{L})) = 0$.

2. If $W_0(\mathcal{L})$ is $\delta$, $\sigma[W_0(\mathcal{L})] = \rho(W_0(\mathcal{L}))$ and $\mu^\ast(I_p(\mathcal{L})) = 0$ then $\mu$ is s. o. a.

**Proof.**

1. Suppose $\mu$ is strong $\sigma$ additive but $\mu^\ast(I_p(\mathcal{L})) = \delta = 0$ then from lemma (4.1) $\exists \rho$ countably additive on $\sigma(\tau W(\mathcal{L}))$ and $\tau W(\mathcal{L})$ regular such that $0 \leq \rho \leq \mu$ and $\rho^\ast(I_p(\mathcal{L})) = \rho(I_p(\mathcal{L})) = 0$ from previous lemma 4.2 $\rho = \gamma$ where $\gamma' \in M_{p}^\ast[W_0(\mathcal{L})]$ then

$$0 \leq \rho - \gamma \leq \mu \Rightarrow 0 \leq \gamma \leq \mu \Rightarrow 0 \leq \gamma \leq \mu$$

and since $\mu$ is s. o. a. then $\gamma = 0$ which is a contradiction to the fact that

$$\rho(I_p(\mathcal{L})) = \gamma(I_p(\mathcal{L})) = a = 0$$

and hence $\mu^\ast(I_p(\mathcal{L})) = 0$.

2. Suppose $W_0(\mathcal{L})$ is $\delta$, $\sigma[W_0(\mathcal{L})] = \rho(W_0(\mathcal{L}))$ and $\mu^\ast(I_p(\mathcal{L})) = 0$. Let $\gamma \in M(\mathcal{L})$, $0 \leq \gamma \leq \mu$ and $\gamma' \in M'[W_0(\mathcal{L})]$ and $\tau$-smooth on $W_0(\mathcal{L})$ then $\gamma' \in M_{p}^\ast[W_0(\mathcal{L})]$ and even $\gamma' \in M_{p}^\ast[W_0(\mathcal{L})]$. So

$$0 \leq \gamma' \leq \mu$$

and therefore $0 \leq \gamma' \leq \mu$ on $\mathcal{A}(W_0(\mathcal{L}))$. Furthermore $0 \leq \gamma' \leq \mu$ and since $\mu^\ast(I_p(\mathcal{L})) = 0$ then $\gamma^\ast(I_p(\mathcal{L})) = 0$ i.e. $\gamma = 0$ i.e. $\mu$ is s. o. a.

**Note.** If $\mathcal{L}$ is $\delta$ and $\sigma(\mathcal{L}) = \rho(\mathcal{L})$ then $W_0(\mathcal{L})$ is $\delta$ and $\sigma[W_0(\mathcal{L})] = \rho(W_0(\mathcal{L}))$ will hold.

**Proposition 4.1.** Let $\mathcal{L}$ be separating and disjunctive if $\mathcal{L}$ is also $\delta$ and $\sigma(\mathcal{L}) = \rho(\mathcal{L})$ then $\mu$ is s. o. a. if $\mu = \mu$ is p. o. a.

**Proof.** $\mu$ is s. o. a. $\Rightarrow \mu^\ast(I_p(\mathcal{L})) = 0 \Rightarrow \mu^\ast(X) = 0; \mu^\ast(X) = 0$ and $\mathcal{L}$ is $\delta$ and $\rho(\mathcal{L}) = \sigma(\mathcal{L}) \Rightarrow \mu$ is p. o. a.

**Proposition 4.2.** If $\mathcal{L}$ is disjunctive then $\mu$ is s. f. a. if and only if $\mu$ is p. f. a.

**Proof.**

1. Suppose $\mu$ is s. f. a. and $\gamma' \in M_p(\mathcal{L})$, $0 \leq \gamma \leq \mu$ then $\gamma' \in M'[W_0(\mathcal{L})]$ and $0 \leq \gamma \leq \mu \Rightarrow \gamma = 0$ by s. f. a. Therefore $\mu$ is p. f. a.

2. Suppose $\mu$ is p. f. a. and $\gamma' \in M'[W_0(\mathcal{L})]$, $0 \leq \gamma \leq \mu$ then $\gamma \in M'(\mathcal{L})$ and $0 \leq \gamma \leq \mu \Rightarrow \gamma = 0$ by purely finitely additive. Therefore $\mu$ is s. f. a.

**Proposition 4.3.** If $\mathcal{L}$ is replete then $\mu$ is s. o. a. if and only if $\mu$ is p. o. a.
PROOF. $L$ replete $\iff X = L^\mu(L) = L^\nu(L)$ then $L = W_o(L)$ and so $\gamma \in M^\nu(L)$ and $\tau$-smooth on $L$ $\iff \gamma' \in M^\nu(W_o(L))$ and $\tau$-smooth or $W_o$ therefore the definitions are equivalent.

THEOREM 4.3. Suppose $L$ is separating, disjunctive and $\sigma(L) = \rho(L)$ then $L$ is replete if and only if for any $\mu \in M^\mu(L)$, $\mu$ is $p$. $\sigma$. a. $\iff \mu$ is $s$. $\sigma$. a.

PROOF.

1. We saw in proposition 4.3 that if $L$ is replete then $p$. $\sigma$. a. $\iff s$. $\sigma$. a.

2. Conversely suppose that $\mu$ is $p$. $\sigma$. a. $\iff s$. $\sigma$. a. for any $\mu \in M^\mu(L)$ but $X \neq I^\mu(L)$. Let $\mu \in I^\mu(L)$ then $\mu$ is $\tau W(L)$ regular and $S(\mu) = \{\mu\}$, $\mu^*(X) = 0$. Now since $\mu^*(X) = 0, L$ is $\delta$ and $\sigma(L) = \rho(L)$ then from theorem 4.1 $\mu$ is purely $\sigma$ additive by assumption; but $\mu$ is $s$. $\sigma$. a. $\iff \mu^*(I^\mu(L)) = 0$ from proposition 4.2; which is a contradiction because $\mu \in M^\mu(L)$ and $\mu\{\mu\} = 1$. Therefore $X = I^\mu(L)$.

DEFINITION 4.3. Let $\mu \in M^\mu(L)$.

1. We say that $\mu$ is $p$. $\tau$ a. if for $\gamma \in M(L)$, $0 \leq \gamma \leq \mu$, and $\gamma L$-tight then $\gamma = 0$.

2. We say that $\mu$ is $s$. $\tau$ a. if for $\gamma' \in M^\gamma(W_o(L))$, $0 \leq \gamma \leq \mu$ on $A(L)$ and $\gamma'$ is $W_o(L)$-tight then $\gamma = 0$.

THEOREM 4.4. Let $L$ be a separating, disjunctive and normal lattice. If $\mu \in M^\mu(L)$ then:

1. $\mu$ is $p$. $\tau$ a. $\iff \mu^*(I^\mu(L) - X) = \mu^*(I^\mu(L))$.

2. $\mu$ is $s$. $\tau$ a. $\iff \mu^*(I^\mu(L) - I^\mu(L)) = \mu^*(I^\mu(L))$.

If we further assume that $L$ is $\delta$ and $\sigma(L) = \rho(L)$ then the converses are true.

PROOF. We will prove only the second proposition and the proof of the first is similar.

2.a) Suppose $\mu$ is $s$. $\tau$ a. but $\mu^*(I^\mu(L) - I^\mu(L)) < \mu^*(I^\mu(L))$, then there exists $G \in [\tau W(L)]$ such that $I^\mu(L) - I^\mu(L) \subset G$ and $\mu^*(G) < \mu^*(I^\mu(L))$. Let $F = I^\mu(L) - G, F \in \tau W(L)$ then $F \subset I^\mu(L)$ and $F$ is $W_o(L)$ compact, for if $F \subset \bigcup_\alpha W_o(L)$ then $F \subset \bigcup_\alpha W_o(L)$. Therefore

$$F \subset \bigcup_\alpha W_o(L) = W(L) \tau L \in L$$

thus $F \subset W_o(L)$ since $F \subset I^\mu(L)$ and $\mu(F) > 0$ since $\mu^*(F) < \mu^*(I^\mu(L))$. Also since $W_o(L)$ is normal and $T_1$ then $F \in \tau W_o(L)$. Now $\mu \in M^\mu(L)$ projects onto $I^\mu(L)$ and $\mu'$ is the projection on $W_o(L)$ and $\mu''$ is the projection on $\tau W_o(L)$. For $E \in A(W_o(L)) \setminus E \tau''(E \cap F) \tau \theta \leq \lambda(E) \leq \mu''(E) = \mu''(E)$ so $0 \leq \lambda \leq \mu'$ on $A(W_o(L))$. Now if $W_o(L) \downarrow \varnothing, L \in L$ then $W_o(L) \cap \varnothing$ and $\lambda[W_o(L)] = \mu''[W_o(L)] \cap F] \to 0$ then

$$\lambda \in M^\mu(W_o(L)).$$

Since $\lambda$ is $\tau$-smooth and $W_o(L)$ is regular. Also $\lambda \in M^\mu[\tau W_o(L)]$ since $\forall \in \varnothing, \lambda[I^\mu(L)] = \mu''(F)$ then

$$\lambda''(F) = \lambda'[\bigcap_\alpha \lambda W_o(L)] = \inf_\alpha \lambda[W_o(L)]$$

$$= \inf \mu''[W_o(L) \cap F] = \mu''(W_o(L) \cap F) = \mu''(F).$$

Therefore

$$\lambda''(F) = \mu''(F) = \lambda[I^\mu(L)] - \lambda[I^\mu(L)] - \varepsilon.$$
Thus
\[ \lambda \in M_0^\ast[W_\alpha(\mathcal{L})] \]
Therefore
\[ \lambda - \gamma' \in M_0^\ast[W_\alpha(\mathcal{L})] \]
so
\[ 0 \leq \gamma' \leq \mu' \text{ on } A[W_\alpha(\mathcal{L})] \text{ and } 0 \leq \gamma \leq \mu \text{ on } A(\mathcal{L}) \]
and \( \gamma' \in M_0^\ast[W_\alpha(\mathcal{L})] \) and \( \lambda - \gamma' = 0 \) contradiction. Hence
\[ \tilde{\mu}(I_R(\mathcal{L}) - I_R^\gamma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L})) \]

2.h) Let \( \gamma' \in M_0^\ast[W_\alpha(\mathcal{L})] \) then \( \gamma \in M_0^\ast(\mathcal{L}) \) also \( \gamma' \in M_0^\ast[W_\alpha(\mathcal{L})] \) because \( \gamma' \) is \( W_\alpha(\mathcal{L}) \)-tight. Now
\[ 0 \leq \gamma' \leq \mu' \text{ on } A[W_\alpha(\mathcal{L})] \Rightarrow 0 \leq \gamma'' \leq \mu'' \text{ on } A[W_\alpha(\mathcal{L})] \]
also \( I_R^\gamma(\mathcal{L}) \) is \( \gamma' \)-measurable since \( \gamma' \in M_0^\ast[W_\alpha(\mathcal{L})] \) then \( \gamma'(I_R(\mathcal{L})) = \gamma(I_R(\mathcal{L})) \) from previous work. Therefore \( \exists F, W_\alpha(\mathcal{L}) \)-compact, \( F \subseteq I_R^\gamma(\mathcal{L}) \) such that
\[ \gamma''(F) = \frac{1}{2}\gamma''(I_R(\mathcal{L})) + \frac{1}{2}\gamma[I_R^\gamma(\mathcal{L})] \]
so
\[ \gamma''(F) \leq \mu''(F) = 0 \text{ since } F \subseteq I_R^\gamma(\mathcal{L}) \]
and since by hypothesis
\[ \tilde{\mu}(I_R(\mathcal{L}) - I_R^\gamma(\mathcal{L})) = \tilde{\mu}(I_R(\mathcal{L})) \]
then
\[ \tilde{\mu}(I_R(\mathcal{L})) = 0 \]
and
\[ \tilde{\mu}(F) = 0 \]
but then
\[ \gamma'(I_R^\gamma(\mathcal{L})) = 0 \Rightarrow \gamma' = 0 \Rightarrow \gamma = 0 \]
therefore \( \mu \) is s. t. a.

REFERENCES