PROLONGATIONS OF F-STRUCTURE TO THE TANGENT BUNDLE OF ORDER 2

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(Received February 21, 1991 and in revised form March 29, 1991)

ABSTRACT. A study of prolongations of F -structure to the tangent bundle of order 2 has been presented.

KEY WORDS AND PHRASES. Prolongations, tangent bundle, integrable, lift, F -structure.

1991 AMS SUBJECT CLASSIFICATION CODE. 53C15

1. INTRODUCTION.

Let F be a nonzero tensor field of type $(1,1)$ and of class c^∞ on an n -dimensional manifold V_n such that [1]

$$F^K + (-)^{K+1}F = 0 \text{ and } F^W + (-)^{W+1}F \neq 0 \quad \text{for } 1 < W < K \quad (1.1)$$

where K is a fixed positive integer greater than 2. Such a structure on V_n is called an F -structure of rank ' r ' and degree K . If the rank of F is constant and $r = r(F)$, then V_n is called an F -structure manifold of degree K (≥ 3). The case when K is odd has been considered in this paper.

Let the operators on V_n be defined as follows [1]:

$$l = (-)^K F^{K-1} \text{ and } m = I + (-)^{K+1} F^{K-1} \quad (1.2)$$

where I denotes the identity operator on V_n .

From the operators defined by (1.2) we have

$$l + m = I \text{ and } l^2 = l; \text{ and } m^2 = m \quad (1.3)$$

For F satisfying (1.1), there exist complementary distributions L and M corresponding to the projection operators l and m respectively.

If rank (F) = constant on V_n then $\dim L = r$ and $\dim M = (n - r)$. We have the following results [1]

$$Fl = lF = F \text{ and } Fm = mF = 0 \quad (1.4a)$$

$$F^{K-1}l = -l \text{ and } F^{K-1}m = 0 \quad (1.4b)$$

2. PROLONGATIONS OF F-STRUCTURE IN THE TANGENT BUNDLE OF ORDER 2.

Let V_n be an n -dimensional differentiable manifold of class c^∞ and $T_p(V_n) = \bigcup_{p \in V_n} T_p(V_n)$ is the tangent bundle over the manifold V_n .

Let us denote $T_s^*(V_n)$, the set of all tensor fields of class c^∞ and of the type (r,s) in V_n and $T(V_n)$ be the tangent bundle over V_n .

Let us introduce an equivalence relation $\sim$ in the set of all differentiable mappings $F: R \rightarrow V_n$ where R is the real line. Let $r \geq 1$ be a fixed integer. If two mappings $F: R \rightarrow V_n$ and $G: R \rightarrow V_n$ satisfy the conditions

$$F^h(0) = G^h(0), \frac{dF^h(0)}{dt} = \frac{dG^h(0)}{dt}, \dots, \frac{dF^r(0)}{dt^r} = \frac{dG^r(0)}{dt^r},$$

the mapping F and G being represented respectively by $X^h = F^h(t)$ and $X^h = G^h(t)$, ($t \in R$) with respect to local coordinates X^h in a coordinate neighborhood $\{U, X^h\}$ containing the point $P = F(0) = G(0)$, then we say that the mapping F is equivalent to G . Each equivalence class determined by the equivalence relation $\sim$ is called an r -jet of V_n and denoted by $J_p^r(F)$. The set of all r -jets of V_n is called the tangent bundle of order r and denoted by $T_r(V_n)$. The tangent bundle $T_2(V_n)$ of order 2 has the natural bundle structure over V_n , its bundle projection $\pi_2: T_2(V_n) \rightarrow V_n$ being defined by $\pi_2(J_p^2(F)) = P$. If we introduce a mapping such that $P = F(0)$, then $T_2(V_n)$ has a bundle structure over $T(V_n)$ with projection π_{12} .

Let us denote $T_2(V_n)$, the second order tangent bundle over V_n and let F^{II} be the second lift of F in $T_2(V_n)$. The second lift F^{II} which belong to $T_s^r(T_2(V_n))$ has component of the form [3]

$$F^{II}: \begin{bmatrix} F_i^h & 0 & 0 \\ y^s \delta_s F_i^h & F_i^h & 0 \\ z^s \delta_s F_i^h + (1/2) y^t y^s \delta_t \delta_s F_i^h & y^s \delta_s F_i^h & F_i^h \end{bmatrix} \quad (2.1)$$

with respect to the induced coordinates in $T_2(V_n)$, F_i^h being local components of F in V_n .

Now we obtain the following results on the second lift of F satisfying (1.1).

For any $F, G \in T_1^1(V_n)$, the following holds [3]:

$$\begin{aligned} (G^{II} F^{II}) X^{II} &= G^{II} (F X^{II}), \\ &= G^{II} (F X)^{II} \\ &= (G(FX))^{II} \\ &= (GF)^{II} X^{II} \quad \text{for every } X \in T_0^1(V_n), \end{aligned} \quad (2.2)$$

therefore we have

$$G^{II} F^{II} = (GF)^{II}$$

If $P(s)$ denote a polynomial of variable s , then we have

$$(P(F))^{II} = P(F^{II}), \text{ where } F \in T_1^1(V_n) \quad (2.3)$$

We have the following theorem:

THEOREM 2.1. The second lift F^{II} defines a F -structure in $T_2(V_n)$ iff F defines a F -structure in V_n .

PROOF. Let F satisfy (1.1) then F defines F -structure in V_n satisfying

$$F^K + (-)^{K+1} F = 0.$$

which in view of equation (2.3) yields

$$(F^{II})^K + (-)^{K+1} F^{II} = 0. \quad (2.4)$$

Therefore F^{II} defines a F -structure in $T_2(V_n)$. The converse can be proved in a similar manner.

THEOREM 2.2. The second lift F^{II} is integrable in $T_2(V_n)$, iff F is integrable in V_n .

PROOF. Let us denote N_{II} and N , the Nijenhuis tensors of F^{II} and F respectively. Then we have [2]

$$N_{II}(X, Y) = (N(X, Y))^{II} \quad (2.5)$$

We know that F -structure is integrable in V_n , iff

$$N(X, Y) = 0,$$

which in view of (2.5) is equivalent to

$$N_{II}(X, Y) = 0. \quad (2.6)$$

Thus F^{II} is integrable, iff F is integrable in V_n .

THEOREM 2.3. The second lift F^{II} of F is partially integrable in $T_2(V_n)$, iff F is integrable in V_n .

PROOF. We know that for F to be partially integrable in V_n , the following holds [2]:

$$N(lX, lY) = 0$$

and

$$N(mX, mY) = 0,$$

which, in view of equation (2.5), takes the form

$$N_{II}(l^{II}X^{II}, l^{II}Y^{II}) = 0$$

and

$$N_{II}(m^{II}X^{II}, m^{II}Y^{II}) = 0. \quad (2.7)$$

where l^{II}, m^{II} are operators in $T_2(V_n)$ which define the distribution L^{II} and M^{II} respectively. Thus equation (2.7) gives the condition for F^{II} to be partially integrable.

The converse follows in a similar manner.

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