INTEGRAL MEANS OF CERTAIN CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper we discuss the following class of functions

$$S_{\lambda}(\alpha, \beta) = \{f(z): \left| \frac{f(z)}{g(z)} - 1 \right| < \beta \left| \lambda \frac{f(z)}{g(z)} + 1 \right|, z \in D\} \text{ where } 0 \leq \lambda \leq 1, 0 < \beta \leq 1, 0 \leq \alpha < 1,$$

and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $D = \{z: |z| < 1\}$, $g(z)$ is a starlike function of order α . A subordination about this class is obtained, the integral means of functions in $S_{\lambda}(\alpha, \beta)$ and some extremal properties are studied.

KEY WORDS AND PHRASES. Analytic function, subordination, integral mean, distortion, coefficient inequality.

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1. INTRODUCTION.

Let A be the class consisting of all functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in $D = \{z: |z| < 1\}$. Owa [1] has introduced the class $\tilde{S}_{\lambda}(\alpha, \beta)$. If $f(z) \in A$ and there exists $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n \in S^*(\alpha)$ ($0 \leq \alpha < 1$) such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \beta \left| \lambda \frac{f(z)}{g(z)} + 1 \right| \quad (0 \leq \lambda \leq 1, 0 < \beta \leq 1, z \in D), \quad (1.1)$$

we say $f(z) \in \tilde{S}_{\lambda}(\alpha, \beta)$. Owa [1] discussed the coefficient estimates of functions in $\tilde{S}_{\lambda}(\alpha, \beta)$. In this paper, we discuss the general case, i.e., the class $S_{\lambda}(\alpha, \beta)$ which is generated by a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*(\alpha).$$

We first gave a subordinate about this class, then we discuss the integral means of functions in $S_{\lambda}(\alpha, \beta)$, from this we can get some extremal properties about $S_{\lambda}(\alpha, \beta)$. We also discuss a subclass of $S_{\lambda}(\alpha, \beta)$.

2. A SUBORDINATION ABOUT $S_{\lambda}(\alpha, \beta)$.

We say that $g(z)$ is subordinate to $f(z)$ if there exists a function $\omega(z)$ analytic in D satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $g(z) = f(\omega(z))$ ($|z| < 1$). This subordination is denoted by $g(z) \prec f(z)$. About the class $S_{\lambda}(\alpha, \beta)$, we have the following:

THEOREM 2.1. If $f(z) \in S_{\lambda}(\alpha, \beta)$, i.e., there exists a function $g(z) \in S^*(\alpha)$ such that the inequality (1.1) holds, then we have

$$\frac{f(z)}{g(z)} \prec \frac{1 + \beta z}{1 - \beta \lambda z} = p_{\beta, \lambda}(z). \quad (2.1)$$

PROOF. Let $p(z) = \frac{f(z)}{g(z)}$, then $p(0) = 1$. Now we divide the proof into three cases.

CASE (a). Let $\lambda \neq 0$, β and λ are not equal to 1 at the same time. Now the inequality (1.1) can be written as $|p(z) - 1| < |\beta\lambda p(z) + \beta|$, that is,

$$|p(z)|^2 - 2\operatorname{Re} p(z) + 1 < \beta^2 \lambda^2 |p(z)|^2 + 2\beta^2 \lambda \operatorname{Re} p(z) + \beta^2. \text{ From this we can get}$$

$$\left| p(z) - \frac{1-\beta}{1+\beta\lambda} - \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2} \right| < \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2}.$$

Because univalent function $p_{\beta,\lambda}(z) = \frac{1+\beta z}{1-\beta\lambda z}$ maps D onto the disk

$$\left\{ w: \left| w - \frac{1-\beta}{1+\beta\lambda} - \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2} \right| < \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2} \right\},$$

so $p(D) \subset p_{\beta,\lambda}(D)$ and $p(0) = p_{\beta,\lambda}(0) = 1$. From the principle of subordination of univalent functions, we have $p(z) \prec p_{\beta,\lambda}(z)$, that is (2.1).

CASE (b). Let $\lambda = 0$. Now the inequality (1.1) becomes $|p(z) - 1| < \beta$.

Because univalent function $p_{\beta,0}(z) = 1 + \beta z$ maps D onto the disk $\{w: |w - 1| < \beta\}$, so $p(D) \subset p_{\beta,0}(D)$ and $p(0) = p_{\beta,0}(0) = 1$. Thus $p(z) \prec p_{\beta,0}(z)$.

CASE (c). Let $\lambda = \beta = 1$. The inequality (1.1) becomes $|p(z) - 1| < |p(z) + 1|$, that is

$\operatorname{Re} p(z) > 0$. Because $p(0) = 1$, so $p(z) \prec \frac{1+z}{1-z} = p_{1,1}(z)$.

Thus for any $0 \leq \lambda \leq 1, 0 < \beta \leq 1$, we have proved (2.1).

3. THE INTEGRAL MEANS OF FUNCTIONS IN $S_\lambda(\alpha, \beta)$.

We first state some lemmas.

LEMMA 3.1 [2]. For any $g, h \in L^1[-\pi, \pi]$, the following statements are equivalent:

(a) For every convex non-decreasing function Φ on $(-\infty, \infty)$,

$$\int_{-\pi}^{\pi} \Phi(g(x)) dx \leq \int_{-\pi}^{\pi} \Phi(h(x)) dx.$$

(b) For every $t \in (-\infty, \infty)$,

$$\int_{-\pi}^{\pi} (g(x) - t)^+ dx \leq \int_{-\pi}^{\pi} (h(x) - t)^+ dx.$$

(c) $g^*(\theta) \leq h^*(\theta)$, $(0 \leq \theta \leq \pi)$.

LEMMA 3.2 [2]. If g, h are real integrable functions on $[-\pi, \pi]$, then $(g + h)^*(\theta) \leq g^*(\theta) + h^*(\theta)$ $(0 \leq \theta \leq \pi)$, with equality holding if and only if g, h are symmetric decreasing arrangement functions.

The definitions of $u^*(x)$ and the symmetric decreasing arrangement function can be found in [2].

LEMMA 3.3 [3]. Let $\Phi(t)$ be a convex increasing function, if $g(z) \prec f(z)$ in D , then

$$\int_{-\pi}^{\pi} \Phi(|g(re^{i\theta})|) d\theta \leq \int_{-\pi}^{\pi} \Phi(|f(re^{i\theta})|) d\theta \quad (0 < r < 1) \quad (3.1)$$

and if $u(z)$ is a harmonic function in D , $v(z) = u(\omega(z))$, where $\omega(z)$ is analytic in D , $\omega(0) = 0$, $|\omega(z)| < 1$, then

$$\int_{-\pi}^{\pi} \Phi(\pm v(re^{i\theta})) d\theta \leq \int_{-\pi}^{\pi} \Phi(\pm u(re^{i\theta})) d\theta \quad (0 < r < 1). \quad (3.2)$$

When $f(z)$ is not a constant, the equality in (3.1) holds if and only if $\omega(z) = e^{i\theta} z$

or $\Phi(u) = a \log u + b$ ($a < 0$).

Let

$$k_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}},$$

it is well known that $k_\alpha(z) \in S^*(\alpha)$. For any $g(z) \in S^*(\alpha)$, we have

$$g(z) = z \exp \left\{ 2(1-\alpha) \int_{|x|=1} \log \frac{1}{1-xz} d\mu(x) \right\},$$

so we can easily obtain

$$\frac{g(z)}{z} \prec \frac{1}{(1-z)^{2(1-\alpha)}} = \frac{k_\alpha(z)}{z}. \quad (3.3)$$

THEOREM 3.1. If $f(z) \in S_{\lambda}(\alpha, \beta)$, $F_x(z) = e^{-ix} k_\alpha(e^{ix}z) \cdot p_{\beta, \lambda}(e^{ix}z)$, $\Phi(t)$ is a convex non-decreasing function on $(-\infty, \infty)$, then

$$\int_{-\pi}^{\pi} \Phi \left(\pm \log \frac{|f(re^{i\theta})|}{r} \right) d\theta \leq \int_{-\pi}^{\pi} \Phi \left(\pm \log \frac{|F_\theta(re^{i\theta})|}{r} \right) d\theta \quad (0 < r < 1). \quad (3.4)$$

For a strictly convex function Φ , the equality holds only for $f(z) = F_x(z)$.

PROOF. From the definition of $S_{\lambda}(\alpha, \beta)$, we know there exists a function $g(z) \in S^*(\alpha)$ such that the inequality (1.1) holds. So we have, from Theorem 2.1

$$p(z) = \frac{f(z)}{g(z)} \prec \frac{1+\beta z}{1-\beta \lambda z} = P_{\beta, \lambda}(z)$$

Thus

$$\int_{-\pi}^{\pi} \Phi(\pm \log |p(re^{i\theta})|) d\theta \leq \int_{-\pi}^{\pi} \Phi(\pm \log |p_{\beta, \lambda}(re^{i\theta})|) d\theta, \quad \text{by Lemma 3.3.}$$

Then from Lemma 3.1 we have

$$(\log |p(re^{i\theta})|)^* \leq (\log |p_{\beta, \lambda}(re^{i\theta})|)^*.$$

On the other hand, because $\frac{f(z)}{z} = p(z) \cdot \frac{g(z)}{z}$, we have, by Lemma 3.2,

$$\left(\log \frac{|f(re^{i\theta})|}{r} \right)^* \leq (\log |p(re^{i\theta})|)^* + \left(\log \frac{|g(re^{i\theta})|}{r} \right)^*$$

Using (3.3) and Lemmas 3.3 and 3.1, we can easily get

$$\left(\log \frac{|g(re^{i\theta})|}{r} \right)^* \leq \left(\log \frac{|k_\alpha(re^{i\theta})|}{r} \right)^*.$$

So we obtain

$$\left(\log \frac{|f(re^{i\theta})|}{r} \right)^* \leq (\log |p_{\beta, \lambda}(re^{i\theta})|)^* + \left(\log \frac{|k_\alpha(re^{i\theta})|}{r} \right)^*.$$

By evaluation we know $\log |p_{\beta, \lambda}(re^{i\theta})|$ and $\log \frac{|k_\alpha(re^{i\theta})|}{r}$ are symmetric decreasing arrangement functions, so again from Lemma 3.2 we have

$$\left(\log \frac{|f(re^{i\theta})|}{r} \right)^* \leq \left(\log \left| p_{\beta, \lambda}(re^{i\theta}) \cdot \frac{k_\alpha(re^{i\theta})}{r} \right| \right)^* = \left(\log \frac{|F_\theta(re^{i\theta})|}{r} \right)^*$$

Finally we obtain, by Lemma 3.1,

$$\int_{-\pi}^{\pi} \Phi \left(\log \frac{|f(re^{i\theta})|}{r} \right) d\theta \leq \int_{-\pi}^{\pi} \Phi \left(\log \frac{|F_\theta(re^{i\theta})|}{r} \right) d\theta$$

We can similarly prove the case of negative sign. The condition of the equality can easily be obtained.

THEOREM 3.2. Let $f(z) \in S_{\lambda}(\alpha, \beta)$, then for $p > 0$ we have

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \leq \int_{-\pi}^{\pi} |k_\alpha(re^{i\theta}) p_{\beta, \lambda}(re^{i\theta})|^p d\theta \quad (0 < r < 1) \quad (3.5)$$

and

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{-p} d\theta \leq \int_{-\pi}^{\pi} |k_{\alpha}(re^{i\theta}) p_{\beta, \lambda}(re^{i\theta})|^{-p} d\theta \quad (0 < r < 1) \quad (3.6)$$

where the equality holds only for $f(z) = \frac{1}{z} k_{\alpha}(xz) p_{\beta, \lambda}(xz)$, $|z| = 1$.

PROOF. We only need let $\Phi(t) = e^{pt}$ in Theorem 3.1.

COROLLARY 3.1. If $f(z) \in S_{\lambda}(\alpha, \beta)$, then we have the following sharp inequality:

$$\frac{r(1-\beta r)}{(1+r)^{2(1-\alpha)}(1+\beta \lambda r)} \leq |f(z)| \leq \frac{r(1+\beta r)}{(1-r)^{2(1-\alpha)}(1-\beta \lambda r)} \quad (|z| = r). \quad (3.7)$$

PROOF. Take p -th root in both sides of (3.5) and (3.6), and let $p \rightarrow \infty$, we can get inequality (3.7).

COROLLARY 3.2. If $f(z) \in S_{\lambda}(\alpha, \beta)$, then we have $f(D) \supset \{w: |w| < d(\alpha, \beta, \lambda)\}$, where

$$d(\alpha, \beta, \lambda) = \frac{1-\beta}{2^{2(1-\alpha)}(1+\beta \lambda)}$$

cannot be replaced by any larger number.

PROOF. We can easily know $f(z)$ is univalent in D from the definition of $S_{\lambda}(\alpha, \beta)$, so

$$\text{dist}(0, \partial f(D)) = \lim_{|z| \rightarrow 1} \inf |f(z)| \geq \lim_{|z| \rightarrow 1} \frac{|z|(1-\beta|z|)}{(1+|z|)^{2(1-\alpha)}(1+\beta \lambda|z|)} = \frac{1-\beta}{2^{2(1-\alpha)}(1+\beta \lambda)}.$$

The sharpness can be seen from the function $\frac{z(1+\beta z)}{(1-z)^{2(1-\alpha)}(1-\beta \lambda z)} \in S_{\lambda}(\alpha, \beta)$.

4. A SUBCLASS $\in S_{\lambda}(\alpha, \beta)$.

Let $g(z) = z$, we obtain a subclass $\in S_{\lambda}(\alpha, \beta)$, we denote it by $S_{\lambda}(\beta)$. Corresponding to (2.1), for the class $S_{\lambda}(\beta)$, we have the following subordination:

$$\frac{f(z)}{z} \prec \frac{1+\beta z}{1-\beta \lambda z} = p_{\beta, \lambda}(z). \quad (4.1)$$

Thus for $S_{\lambda}(\beta)$ we have

THEOREM 4.1. Let $f(z) \in S_{\lambda}(\beta)$, $\Phi(t)$ is a convex non-decreasing function on $(-\infty, \infty)$, then

$$\int_{-\pi}^{\pi} \Phi\left(\pm \log \frac{|f(re^{i\theta})|}{r}\right) d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\pm \log \left|\frac{1+\beta re^{i\theta}}{1-\beta \lambda re^{i\theta}}\right|\right) d\theta \quad (0 < r < 1). \quad (4.2)$$

For a strictly convex function Φ , the equality holds only for function $f(z) = zp_{\beta, \lambda}(xz)$, $|z| = 1$

If we use subordination (4.1) and Lemma 3.3, we can obtain the following:

THEOREM 4.2. Let $f(z) \in S_{\lambda}(\beta)$, $\Phi(t)$ is a convex non-decreasing function on $(-\infty, \infty)$, then

$$(a) \quad \int_{-\pi}^{\pi} \Phi\left(\frac{|f(re^{i\theta})|}{r}\right) d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\left|\frac{1+\beta re^{i\theta}}{1-\beta \lambda re^{i\theta}}\right|\right) d\theta, \quad (4.3)$$

$$(b) \quad \int_{-\pi}^{\pi} \Phi\left(\left|\log \frac{f(re^{i\theta})}{re^{i\theta}}\right|\right) d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\left|\log \frac{1+\beta re^{i\theta}}{1-\beta \lambda re^{i\theta}}\right|\right) d\theta, \quad (4.4)$$

$$(c) \quad \int_{-\pi}^{\pi} \Phi\left(\pm \arg \frac{f(re^{i\theta})}{re^{i\theta}}\right) d\theta \leq \int_{-\pi}^{\pi} \Phi\left(\arg \frac{1+\beta re^{i\theta}}{1-\beta \lambda re^{i\theta}}\right) d\theta. \quad (4.5)$$

For a strictly convex function Φ , the equality holds only for $f(z) = zp_{\beta, \lambda}(xz)$, $|z| = 1$. From (4.5) we obtain the rotation theorem of $S_{\lambda}(\beta)$.

COROLLARY 4.1. Let $f(z) \in S_{\lambda}(\beta)$, then for $|z| = r < 1$ we have the following sharp inequality:

$$\left|\arg \frac{f(z)}{z}\right| \leq \arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2r^2}.$$

PROOF. If we take

$$\Phi(t) = \begin{cases} t^{2n} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

in (4.5), we have

$$\int_0^{2\pi} \left| \arg + \frac{f(re^{i\theta})}{re^{i\theta}} \right|^{2n} d\theta \leq \int_0^{2\pi} \left| \arg + \frac{1 + \beta re^{i\theta}}{1 - \beta \lambda re^{i\theta}} \right|^{2n} d\theta.$$

Take the $2n$ -th root in both sides of this inequality and let $n \rightarrow \infty$, we get

$$\max_{-\pi \leq \theta \leq \pi} \arg + \frac{f(re^{i\theta})}{re^{i\theta}} \leq \max_{-\pi \leq \theta \leq \pi} \arg + \frac{1 + \beta re^{i\theta}}{1 - \beta \lambda re^{i\theta}} = \arcsin \frac{\beta(1+\lambda)r}{1 + \lambda\beta^2 r^2}.$$

This implies

$$\arg + \frac{f(re^{i\theta})}{re^{i\theta}} \leq \arcsin \frac{\beta(1+\lambda)r}{1 + \lambda\beta^2 r^2}.$$

Similarly, we have

$$\arg - \frac{f(re^{i\theta})}{re^{i\theta}} \leq \arcsin \frac{\beta(1+\lambda)r}{1 + \lambda\beta^2 r^2}.$$

So for any $f(z) \in S_\lambda(\beta)$, we have

$$\left| \arg \frac{f(z)}{z} \right| \leq \arcsin \frac{\beta(1+\lambda)r}{1 + \lambda\beta^2 r^2} \quad (|z| = r < 1),$$

where the equality holds only for $f(z) = zp_{\beta, \lambda}(xz)$, $|x| = 1$. The proof of Corollary 4.1 is complete.

From the univalence of $f(z)$ we know $\frac{f(z)}{z} \neq 0$, so we can define a single-valued and analytic branch of $\log \frac{f(z)}{z}$. Let

$$g(z) = \log \frac{f(z)}{z} = \sum_{n=1}^{\infty} \gamma_n z^n,$$

then we have:

COROLLARY 4.2. Let $f(z) \in S_\lambda(\beta)$, then we have

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{(\beta^n \lambda^n + (-1)^{n-1} \beta^n)^2}{n^2}, \quad (4.6)$$

where the inequality holds only for $f(z) = zp_{\beta, \lambda}(xz)$, $|x| = 1$.

PROOF. Let

$$G(z) = \log \frac{1 + \beta z}{1 - \beta \lambda z} = \sum_{n=1}^{\infty} \frac{\beta^n \lambda^n + (-1)^{n-1} \beta^n}{n} z^n.$$

Take $\Phi(t) = t^2$ in (4.4), we have

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^2 d\theta,$$

that is,

$$\sum_{n=1}^{\infty} |\lambda_n|^2 r^{2n} \leq \sum_{n=1}^{\infty} \frac{(\beta^n \lambda^n + (-1)^{n-1} \beta^n)^2}{n^2} r^{2n},$$

let $r \rightarrow 1$, we obtain the inequality we need to prove.

REMARK. Let $\lambda = \beta = 1$ in Corollary 4.2, that is $f(z) \in S_1(1)$, i.e., $\operatorname{Re}(f(z)/z) > 0$.

Inequality (4.6) becomes

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2} = \frac{\pi^2}{2}.$$

This inequality is sharp.

Finally, we consider the initial coefficients of $f(z) \in S_\lambda(\beta)$.

LEMMA 4.1. If $f'(0) = F'(0) = 1$ and they satisfy the following equality

$$\frac{(1-\beta) - (1+\beta\lambda)f(z)/z}{(1-\beta\lambda)f(z)/z - (1+\beta)} = \frac{F(z)}{z}, \quad (4.7)$$

then $f(z) \in S_\lambda(\beta)$ if and only if $F(z) \in S_1(1)$, i.e., $\operatorname{Re}(F(z)/z) > 0$.

PROOF. Let $f(z) \in S_\lambda(\beta)$, then $p(z) = \frac{f(z)}{z} \prec \frac{1+\beta z}{1-\beta\lambda z} = p_{\beta,\lambda}(z)$, so $p(D) \subset p_{\beta,\lambda}(D) = D_1$ where D_1 is a disk which diameter is $(\frac{1-\beta}{1+\beta\lambda}, \frac{1+\beta}{1-\beta\lambda})$, thus

$$(1-\beta\lambda)p(z) - (1+\beta) \neq 0.$$

From this we know $F(z)$ is analytic in D . And $F(0) = 0$ because of $p(0) = 1$. On the other hand, the function

$$\frac{(1-\beta) - (1+\beta\lambda)w}{(1-\beta\lambda)w - (1+\beta)}$$

maps D_1 onto the right half plane, so we have $\operatorname{Re}(F(z)/z) > 0$ ($z \in D$), i.e., $F(z) \in S_1(1)$.

We can prove the opposite result similarly.

THEOREM 4.3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\lambda(\beta)$, then for real number μ we have the sharp estimates:

$$|a_2| \leq \beta(1+\lambda) \quad (4.8)$$

$$|a_3 - \mu a_2^2| \leq \begin{cases} \beta(1+\lambda)(\beta\lambda - \mu(\beta + \beta\lambda)), & \mu \leq -\frac{1-\beta\lambda}{\beta + \beta\lambda}, \end{cases} \quad (4.9)$$

$$|a_3 - \mu a_2^2| \leq \begin{cases} \beta(1+\lambda), & -\frac{1-\beta\lambda}{\beta + \beta\lambda} < \mu < \frac{1+\beta\lambda}{\beta + \beta\lambda}, \end{cases} \quad (4.10)$$

$$|a_3 - \mu a_2^2| \leq \begin{cases} \beta(1+\lambda)(\mu(\beta + \beta\lambda) - \beta\lambda), & \mu > \frac{1+\beta\lambda}{\beta + \beta\lambda}. \end{cases} \quad (4.11)$$

PROOF. Because $f(z) \in S_\lambda(\beta)$, then $F(z)$ defined by (4.7) belongs to $S_1(1)$, i.e., $\operatorname{Re}(F(z)/z) > 0$, so there exists an analytic function $p(z)$ satisfying $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $\operatorname{Re} p(z) > 0$ such that

$$\frac{F(z)}{z} = p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Substituting it into (4.7) and comparing the coefficients of both sides of (4.7), we have

$$a_2 = \frac{1}{2}(\beta + \beta\lambda)p_1, \quad a_3 - \mu a_2^2 = \frac{1}{2}(\beta + \beta\lambda) \left\{ p_2 - \frac{1}{2}((1-\beta\lambda) + \mu(\beta + \beta\lambda)) p_1^2 \right\}.$$

It is well known that $|p_n| \leq 2$ ($n = 1, 2, \dots$)

$$|p_2 - \mu p_1^2| \leq \begin{cases} 2(1-2\mu), & \mu \leq 0 \\ 2, & 0 < \mu < 1 \\ 2(2\mu-1), & \mu \geq 1 \end{cases}.$$

So we proved the results. Its easy to know the function $f(z) = \frac{z(1+\beta ez)}{1-\beta\lambda ez}$, ($|e| = 1$) attains the equalities in (4.8), (4.9) and (4.11), and the function $f(z) = \frac{z(1+\beta ez^2)}{1-\beta\lambda ez^2}$, ($|e| = 1$) attains the inequality in (4.10).

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