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SOME FIXED POINT THEOREMS FOR COMPATIBLE MAPS

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ABSTRACT. A collection of fixed point theorems is generalized by replacing hypothesized commutativity or weak commutativity of functions involved by compatibility.

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1. INTRODUCTION. The last two decades have produced a spate of articles which propose generalizations and/or extensions of the Banach Contraction Principle, which Principle states that a contraction f of a complete metric space (X,d) has a unique fixed point. Typical approaches have been either to vary the contraction requirement that $d(fx, fy) \leq r\ d(x, y)$ for some $r \in (0,1)$ and all $x,y \in X$, or to introduce more functions with conditions appended. For example, in 1976 the following result appeared:

THEOREM 1.1. [1] Let f and g be commuting (gf=fg) self maps of a complete metric space (X,d) such that $f(X) \subseteq g(X)$ and g is continuous. If $\exists\ r \in (0,1)$ such that $d(fx, fy) \leq r\ d(gx, gy)$ for $x,y \in X$, then f and g have a unique common fixed point $a \in X$ (i.e., fa=ga=a).

The above theorem and article promoted commutative maps as a tool for generalizing. Subsequently, a variety of variations and generalizations of Theorem 1 which utilized the commuting map concept appeared (See, e.g., [2, 3, 4, 5, 6, 7,]) . In 1982, Sessa [8] introduced a generalization of the commuting map concept by saying that maps $f,g:(X,d)\to(X,d)$ are weakly commutative iff $d(fgx, gfx) \leq d(fx, gx)$ for $x \in X$. In response, variations on Banach and Theorem 1. appeared in terms of "weakly commuting pairs f,g" – see, e.g., [9], [10]. Then, in 1986, the first author introduced the concept of compatibility.

DEFINITION 1.1. ([11]) Self maps f and g of a metric space (X,d) are compatible iff whenever $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to t \in X$, then $d(fgx_n, gfx_n) \to 0$.

Clearly, commuting mappings are weakly commuting and weakly commuting pairs are compatible; examples in [8] and [11] show that neither converse is true. Articles already in print demonstrate that known results can be generalized by using compatibility in lieu of commutativity or weak commutativity. We refer the reader to [11, 12, 13, 14, 15 ,16, 17]; in particular , we note [17] in which Rhoades, Park, and Moon obtain a very general fixed point theorem by using Meir-Keeler type contraction maps in conjuction with compatibility.

The purpose of this paper is to further demonstrate the effectiveness of the compatible map concept as a means of generalizing. We shall show that an appreciable number of fixed point and coincidence theorems can be improved by substituting compatibility for commutativity or weak commutativity. Such an effort seems to be in order, indeed, called for, since – as the reader will see – the method of attack for one theorem is typically very similar to that for another theorem. The approach becomes "standard" because the definition of compatibility and one proposition regarding compatibility are the only tools needed. The proposition we need is Proposition 2.2. in [11].

PROPOSITION 1.1. ([11]) Let f and g be compatible self maps of a metric space (X,d).

1. If $f(t)=g(t)$, then $fg(t)=gf(t)$.
2. Suppose that $\lim_n f(x_n) = \lim_n g(x_n) = t$ for some $t \in X$ and $x_n \in X$.
   (a) If f is continuous at t, $\lim_n gf(x_n) = f(t)$.
   (b) If f and g are continous at t, then $f(t) = g(t)$ and $fg(t) = gf(t)$.

2. GENERALIZATIONS VIA COMPATIBILITY.

We shall now state generalizations of published results, generalizations obtained in the main by replacing the hypothesised commutativity or weak commutativity with compatibility. Proofs of some of these results will be given in relative detail so as to demonstrate techniques involved. Of course, in most instances goodly portions of the proofs of results being generalized will pertain and will be appealed to so as to avoid repetition.

We have taken care to not to duplicate results already in the literature, such as the general theorem of Rhoades, Park, and Bae, and Moon.

The first theorem is a generalization of Theorem 1. in [18], a 1986 paper by Diviccara, Sessa and Fisher. We substitute compatibility for weak commutativity in the hypothesis.

THEOREM 2.1. Let S,T, and I be selfmaps of complete metric space (X,d) such that for $x,y \in X$ either

(a) $d(Sx, Ty) \leq (a\, d(Ix, Sx)\, d(Iy, Ty) + b\, d(Ix, Ty)\, d(Iy, Sx))\, D$, where
$D = (d(Ix, Sx) + d(Iy, Ty))^{-1}$
if $d(Ix, Sx) + d(Iy, Ty) \neq 0$, where $1 < a < 2$ and $b \geq 0$, or

(b) $d(Sx, Ty) = 0$ if $d(Ix, Sx) + d(Iy, Ty) = 0$.

Suppose $S(X) \cup T(X) \subseteq I(X)$. If either I is continuous and compatible with one of S,T, or one of S or T is continuous and compatible with I, then I, S, and T have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and I.

PROOF. The argument in the proof of Theorem [18] on page 278 pertains and we have a sequence $\{x_n\}$ and $w \in X$ such that

$$(*) \quad Ix_n,\ Sx_{2n},\ Tx_{2n-1} \to w.$$

We first consider the case (i) $d_n \neq 0$, where $d_{2n-1} = d(Tx_{2n-1}, Sx_{2n})$ and $d_{2n} = d(Sx_{2n}, Tx_{2n+1})$.

Now assume that I is continuous and compatible with S. Then ISx_{2n}, $IIx_{2n} \to Iw$ by continuity, and $SIx_{2n} \to Iw$ by Proposition (1.1)2(a) since I and S are compatible and I continuous. We assert that Iw=Tw. Otherwise, (a) in the hypothesis implies

$$d(SIx_{2n},Tw) \leq (\, a\, d(IIx_{2n}, SIx_{2n})\, d(Iw, Tw) + b\, d(IIx_{2n}, Tw)\, d(Iw, SIx_{2n}))\, D^{-1}, \text{ where}$$
$$D = d(Iw, Sw) + d(Iw, Tw).$$

But as $n \to \infty$ we obtain $d(Iw,Tw) \leq 0$, a contradiction. Thus, Tw=Iw. The argument given in the third paragraph of page 279 in [18] shows that, in fact, Iw=Sw=Tw.

The case in which I is continuous and compatible with T follows from the above because of the symmetric roles of S and T; i.e., Iw=Sw=Tw in this case also.

Next, suppose that S is continuous and compatible with I. Then (*) above and Proposition(1.1)2(a) imply that SIx_{2n}, $SSx_{2n} \to Sw$ and $ISx_{2n} \to Sw$. Since $S(X) \subseteq I(X)$, there exists $w' \in X$ such that $Iw'=Sw$. In fact, the line of reasoning at the bottom of page 279 and top of page 280 is valid for us because the above sequences do converge to Sw, and we have $Iw'=Sw'=Tw'=Sw$. As above, we can appeal to "symmetry" to conclude that Iz=Sz=Tz for some z when T is continuous and compatible with I.

We have considered all possibilities to show that Iw=Sw=Tw for some $w \in X$ when $d_n \neq 0$. The case in which $d_n=0$ for some n is covered in (ii) and (iii) on page 280 and holds for us. Thus, in any case, Iw=Sw=Tw fox some $w \in X$.

As we now show, Iw is a common fixed point of I, S, and T. Note that the argument given depends on compatibility without any reference to continuity. If I and S are compatible, then Tw=Iw=Sw and Proposition (1.1) 1. imply that SSw = SIw = ISw = IIw. But then d(IIw, SIw) + d(Iw, Tw) = 0, so that d(SIw, Tw) = 0 by (b) of the hypothesis. Therefore, Iw=Tw=SIw=IIw, and z=Iw is a common fixed point of I and S. Moreover, Tz = z. For if not, (a) of the hypothesis yields d(z,Tz) = d(Sz, Tz) $\leq$ (a d(Iz,Sz) d(Iz,Tz) + b d(Iz,Tz) d(Iz,Sz)) $(d(Iz,Sz) + d(Iz,Tz))^{-1} = 0$; i.e., $d(z,Tz) \leq 0$ – a contradiction. Thus, z=Iz=Sz=Tz. The other case, namely, I and T compatible, follows in a similar fashion.

We have shown that, in any case, I, S, and T have a common fixed point. The uniqueness assertions follow immediately from (b) of the hypothesis. □

The next theorem generalizes Theorem 1. [19] of Imdad, Kahn, and Sessa by replacing the weakly commuting requirement of the hypothesis by compatibility. Note that our approach simplifies the argument give in [19] on pages 31-32.

THEOREM 2.2. Let X be a uniformly convex Banach space and K a nonempty closed subset of X. Let A, S, and T be self maps of K satisfying:

(i) S and T are continuous, and $A(K) \subseteq S(K) \cup T(K)$.

(ii) {A,S } and {A,T} are compatible pairs on K.

(iii) There exists an upper semi-continuous function $f:\mathbb{R}_+^5 \to \mathbb{R}_+$ which is non-decreasing in each coordinate variable such that for any $x,y \in K$:
$\|Ax - Ay\| \leq f(\|Sx - Ty\|, \|Sx - Ax\|, \|Sx - Ay\|, \|Ty - Ax\|, \|Ty - Ay\|)$,
where f also satisfies:

(iv) for $t > 0$, $f(t, t, 0, \alpha t, t) \leq \beta t$ and $f(t, t, \alpha t, 0, t) \leq \beta t$ where $\beta < 1$ for $\alpha < 2$, and $\beta = 1$ for $\alpha = 2$, $\alpha, \beta \in \mathbb{R}_+$,

(v) $f(t, 0, t, t, 0) < t$ for $t > 0$.

(a) Then there exists a point $u \in K$ such that u is the unique common fixed point of A, S, and T, and

(b) for any $x_0 \in K$, the sequence $\{Ax_n\}$ defined by $Tx_{2n} = Ax_{2n-1}$ and $Sx_{2n+1} = Ax_{2n}$ for n=0, 1, 2, ... , converges strongly to u.

PROOF. One follows the proof of Imdad, Khan, and Sessa ([19]) through page 31, line 11, and we thus have $Ax_n \to u$, $Sx_{2n+1} \to u$, and $Tx_{2n} \to u$. Since T is continuous, $TAx_{2n} \to Tu$. But A and T are also compatible, so Proposition (1.1)2(a) implies that $ATx_{2n} \to Tu$. Similarly, since S and A are compatible and S is continuous, $SAx_{2n+1} \to Su$ and $ASx_{2n+1} \to Su$.

Suppose $Su \neq Tu$. From (iii) in the hypothesis, $\|ASx_{2n+1} - ATx_{2n}\| \leq$

$f(\|SSx_{2n+1} - TTx_{2n}\|, \|SSx_{2n+1} - ASx_{2n+1}\|, \|SSx_{2n+1} - ATx_{2n}\|, \|TTx_{2n} - ASx_{2n+1}\|,$
$\|TTx_{2n} - ATx_{2n}\|)$.

Taking the limit as $n \to \infty$ yields $\|Su - Tu\| \leq f(\|Su - Tu\|, 0, \|Su - Tu\|, \|Su - Tu\|, 0) <$ $\|Su - Tu\|$, by (v). This contradiction demands that $Su = Tu$.

Likewise, $Su = Au$. For suppose $Su \neq Au$. From (iii), $\|ASx_{2n+1} - Au\| \leq$

$f(\|SSx_{2n+1} - Tu\|, \|SSx_{2n+1} - ASx_{2n+1}\|, \|SSx_{2n+1} - Au\|, \|Tu - ASx_{2n+1}\|, \|Tu - Au\|)$.

As $n \to \infty$ we obtain,

$$\begin{aligned}\|Su - Au\| &\leq f(\|Su - Tu\|, \|Su - Su\|, \|Su - Au\|, \|Tu - Su\|, \|Tu - Au\|) = \\ &= f(0, 0, \|Su - Au\|, 0, \|Tu - Au\|) \\ &\leq f(\|Su - Au\|, \|Su - Au\|, \|Su - Au\|, 0, \|Su - Au\|) \\ &< \|Su - Au\|, \text{ a contradiction.}\end{aligned}$$

We have, $Au = Su = Tu$. The remainder of the proof is the same as that in [19], beginning on the second line from the bottom of page 32 and continuing to middle of page 33, the end of the proof. □

Our next theorem generalizes Theorem 1. ([20]) of Devi Prasad by relaxing the requirement that hf=fh and gh=hg by merely requiring that each of the pairs f,h and g,h be compatible.

THEOREM 2.3. Let f, g, and h be self mappings of a complete metric space (X,d) which satisfy: $f(X) \cup g(X) \subseteq h(X)$, f and h are compatible and g and h are compatible. Suppose further that

(i) $d(fx, gy))^2 \leq \phi(d(hx, fx)\, d(hy, gy), d(hx, gy)\, d(hy, fx), d(hx, fx)\, d(hx, gy), d(hy, fx)\, d(hy, gy))$

for any $x,y \in X$, where $\phi: \mathbb{R}_+^4 \to \mathbb{R}_+$ is upper semi-continuous and nondecreasing in each coordinate variable and satisfies $\phi(t, t, a_1 t, a_2 t) < t$ for any $t>0$, where $a_i \in \{0,1,2\}$ with $a_1 + a_2 = 2$. If h is continuous, then f, g, and h have a unique common fixed point.

PROOF. Follow the proof of Prasad to the bottom of page 1074. Then we have $\{fx_{2n}\}$, $\{gx_{2n+1}\}$, and $\{hx_n\}$ converging to u. Since h is continuous, $h^2x_n \to hu$ and $hfx_{2n} \to hu$, and since h and f are also compatible, $fhx_{2n} \to hu$, by Proposition(1.1)2(a). Similarly, the continuity of h and the compatibility of h and g imply that $hgx_{2n+1} \to hu$ and $ghx_{2n+1} \to hu$.

Now (i) implies:

$$\begin{aligned}d(fhx_{2n}, gu)^2 \leq \phi(&d(hhx_{2n}, fhx_{2n})\, d(hu, gu), d(hhx_{2n}, gu)\, d(hu, fhx_{2n}), \\ &d(hhx_{2n}, fhx_{2n})\, d(hhx_{2n}, gu), d(hu, fhx_{2n})\, d(hu, gu)).\end{aligned}$$

Taking the limit as $n\to\infty$ yields: $d(hu, gu)^2 \leq \phi(0, 0, 0, 0) = 0$. Therefore, $hu = gu$.

Appeal to (i) again to obtain:

$$d(fu, ghx_{2n+1})^2 \leq \phi\,(\, d(hu, fu)\, d(hhx_{2n+1}, ghx_{2n+1})\,,\, d(hu, ghx_{2n+1})\, d(hhx_{2n+1}, fu)\,,$$
$$d(hu, fu)\, d(hu, ghx_{2n+1})\,,\, d(hhx_{2n+1}, fu)\, d(hhx_{2n+1}, ghx_{2n+1})\,).$$

As $n\to\infty$ we obtain, $d(fu, hu)^2 \leq \phi(0, 0, 0, 0) = 0$. Thus $fu = hu$.

The remainder of the proof is the same as in the proof of Theorem 1. of Prasad. □

The next theorem is a generalization of a Theorem 1. in [21] by S. L. Singh on L-spaces. L-spaces utilize semi-metrics d (See [21]). We extend our definition of compatibility to L-spaces by saying that self maps P and Q of an L-space $(X,\to)$ are compatible relative to a semimetric d on X iff whenever $\{x_n\}$ is a sequence in X such that $Px_n\to t$ and $Qx_n\to t$ for some $t \in X$, then $d(PQx_n, QPx_n) \to 0$. Also note that in a separated L-space d is continuous.

THEOREM 2.4. Let $(X,\to)$ be a separated L-space which is d-complete for a semimetric d. Let P, Q, T be continuous selfmaps of $(X,\to)$ such that the pairs P,T and Q,T are each compatible relative to d and satisfy $P(X) \cup Q(X) \subseteq T(X)$. If there exists $h \in (0,1)$ such that for all $x,y \in X$:

$$d(Px, Qy) \leq h \max\{\, d(Px, Tx), d(Qy, Ty), d(Tx, Ty),\}$$

then P, Q, and T have a unique common fixed point.

PROOF. The proof of Theorem 1. in [21] up to the bottom of page 92 is valid under our hypothesis. We thus have, $Tx_n\to z$, $Px_{2n}\to z$, and $Qx_{2n+1}\to z$. The continuity of T, P, Q and of d, in conjunction with the compatibility of the T and P and of T and Q imply that $Pz=Tz$ and $Qz=Tz$. Therefore, by compatibility (let $x_n = z$ for all n in the definition), $PTz = TPz = TTz = TQz = QTz = PQz = QPz = QQz$. But then

$$d(PQz, Qz) \leq h \max\{d(PQz, TQz), d(Qz, Tz), d(TQz, Tz)\}$$
$$= h \max\{0, 0, d(PQz, Qz)\},$$

so that $PQz = Qz$. By the above equalities we Qz is a common fixed point of P, Q, and T. Uniqueness follows immediately from the contractive definition. □

In the above proof we veritably showed that two compatible self maps of a separated L-space commute at coincidence points of the maps. This fact is noted for metric spaces in Proposition (1.1) 1. However, Proposition (1.1)2.(b) says that if E and F are compatible <u>and</u> continuous self maps of a metric space and Ex_n, $Fx_n\to t$, then $Et=Ft$ and $EFt=FEt$. The proof of the following theorem, which is a generalization of Theorem 2. in [22] by Yeh, appeals to this fact. We again generalize by replacing the hypothesised commutativity of pairs of maps by hypothesising compatibility for the corresponding pairs.

THEOREM 2.5. Let E, F, and T be continuous self maps of a complete metric space (X,d) such that E,T and F,T are compatible, and that $E(X) \cup F(X) \subseteq T(X)$. Suppose that

$$d(Ex, Fy) \leq a(\, d(Tx, Ty))\, d(Tx,Ty) + b(d(Tx, Ty))[\, d(Tx, Ex) + d(Ty, Fy)] +$$
$$c(d(Tx, Ty))\, [\, d(Tx, Fy) + d(Ty, Ex)])$$

for all $x,y \in X$, $x \neq y$, where a, b, and c are mappings from $\mathbb{R}_+$ into $[0, 1)$ satisfying the following: If $A = a + 2b + 2c$ where $0 < A(t) < 1$ for $t \in \mathbb{R}_+$, and $\{t_n\}$ is a monotone increasing sequence in $\mathbb{R}_+$ for which $A(t_n)\to 1$ as $n\to\infty$, then $t_n\to 0$ as $n\to\infty$. Then E, F, and T have a unique common fixed point.

PROOF. Proceed as in the proof of Theorem 2. of Yeh until line 5 of page 119. We have: Tx_n, Ex_{2n}, $Fx_{2n+1} \to x \in X$. Since $Tx_{2n}, Ex_{2n} \to x$ and the continuous functions E and T are compatible, $Ex = Tx$ and $ETx = TEx$ by Proposition(1.1)2.(b.). Similarly, $Fx = Tx$ and $FTx = TFx$. Thus, $T(Tx) = T(Ex) = E(Tx) = E(Ex) = T(Ex) = F(Tx) = F(Ex) = F(Fx)$.

The remainder of the proof is as in [22]. □

In [23], Diviccaro, Fisher, and Sessa prove a common fixed point theorem of the "Gregus" type. However, as was communicated to us by Sessa, a very recent paper (1991) by Davies ([24]) subsumes the "Gregus" type theorem in [23]. We now appreciably generalize Davies' result – Theorem 1. in [24] – by replacing the nonexpansive requirement on the linear map I by continuity, and the weakly commuting hypothesis by compatibility.

THEOREM 2.6. Let I and T be compatible self maps of C, a closed convex subset of a Banach space X, satisfying :

$$\|Tx - Ty\| \leq \alpha \|Ix - Iy\| + \beta \max\{\|Tx - Ix\|, \|Ty - Iy\|\} + \\ + \gamma \max\{\|Ix - Iy\|, \|Tx - Ix\|, \|Ty - Iy\|\}$$

for $x,y \in C$, where $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma = 1$. If I is linear and continuous in C and $T(C) \subseteq I(C)$, then T and I have a unique common fixed point w and T is continuous at w.

PROOF. Define $K_n = \{ x \in C : \|Tx - Ix\| \leq 1/n \}$ for all $n \in N$, the set of positive integers. The proof in [24] holds for our hypothesis through to (13), page 240, where we have $\{w\} = A = \cap\{cl(I(K_n)) : n \in N\}$ and we use cl to denote "closure". Since $w \in A$, for each $n \in N$ $\exists$ $y_n \in I(K_n)$ such that $d(y_n, w) < 1/n$. Then $\exists$ $v_n \in K_n$ such that $y_n = Iv_n$; thus $d(Iv_n, w) < 1/n$ and we infer that $Iv_n \to w$. But $v_n \in K_n$ for $n \in N$, so that $\|Tv_n - Iv_n\| < 1/n$ and we also have $Tv_n \to w$. Since I is continuous, $ITv_n \to Iw$ and $IIv_n \to Iw$. Moreover, $TIv_n \to Iw$ by Proposition(1.1), since I and T are compatible and I is continuous. Now by hypothesis,

$$\|TIv_n - Tw\| \leq \alpha \|IIv_n - Iw\| + \beta \max\{\|TIv_n - IIv_n\|, \|Tw - Iw\|\} + \\ \gamma \max\{\|IIv_n - Iw\|, \|TIv_n - IIv_n\|, \|Tw - Iw\|\}$$

for $n \in N$. As $n \to \infty$ we obtain:

$$\|Iw - Tw\| \leq 0 + \beta \|Tw - Iw\| + \gamma \|Tw - Iw\| = (\beta+\gamma) \|Tw - Iw\|.$$

Therefore, since $(\beta+\gamma) < 1$ by hypothesis, $Iw = Tw$. Moreover,

$$\|Tv_n - Tw\| \leq \alpha \|Iv_n - Iw\| + \beta \max\{\|Tv_n - Iv_n\|, \|Tw - Iw\|\} + \\ \gamma \max\{\|Iv_n - Iw\|, \|Tv_n - Iv_n\|, \|Tw - Iw\|\}$$

for $n \in N$. Taking the limit as $n \to \infty$ yields:

$$\|w - Tw\| \leq \alpha \|w - Tw\| + 0 + \gamma \|w - Tw\| = (\alpha+\gamma) \|w - Tw\|.$$

As above, since $(\alpha+\gamma) < 1$, we conclude that $w = Tw$ and we have $w = Tw = Iw$.

That w is that unique <u>common</u> fixed point of I and T follows from the fact that any common fixed point of I and T is in A, and A is a singleton. However, Davies appeals to the nonexpansiveness of I to prove T continuous at w. Since we are only asssuming that I is continuous, we proceed as follows.

Let $x_n \to w$. Since I is continuous, $Ix_n \to Iw = Tw$. Now by hypothesis, using $Iw = Tw$,

$$\|Tx_n - Tw\| \leq \alpha \|Ix_n - Iw\| + \beta \max\{\|Tx_n - Ix_n\|, 0\} + \\ \gamma \max\{\|Ix_n - Iw\|, \|Tx_n - Ix_n\|, 0\}.$$

Since $\|Tx_n - Ix_n\| \leq \|Tx_n - Tw\| + \|Tw - Ix_n\| = \|Tx_n - Tw\| + \|Iw - Ix_n\|$, we then have

$$\|Tx_n - Tw\| \leq \alpha \|Ix_n - Iw\| + (\beta+\gamma)(\|Tx_n - Tw\| + \|Iw - Ix_n\|), \text{ so}$$

$$\|Tx_n - Tw\| \leq (1/\alpha)\|Ix_n - Iw\|,$$

for $n \in N$. Therefore, since $Ix_n \to Iw$, $Tx_n \to Tw$, as desired. □

The next Theorem is a generalization of Theorem 3. in [25], a paper published in 1986 by Fisher and Sessa. We generalize by substituting compatibility for weak commutativity.

THEOREM 2.7. Let $\{S,I\}$ and $\{T,J\}$ be two pairs of compatible self maps of a complete metric space (X,d) such that

$$d(Sx, Ty) \leq g(\, d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\,)$$

for any $x,y \in X$, where $g: \mathbb{R}^3_+ \to \mathbb{R}_+$, is continuous, and satisfies:

(i) $g(\,1, 1, 1\,) = h < 1$, and

(ii) whenever $u,v \geq 0$ and either $u \leq g(u, v, v)$, $u \leq g(v, u, v)$, or $u \leq g(\,v, v, u)$, then $u \leq hv$.

If $T(X) \subseteq I(X)$, $S(X) \subseteq J(X)$, and if one of I, J, S, or T is continuous, then I, J, S, and T have a unique common fixed point z. Further, z is the unique common fixed point of I and S and of J and T.

PROOF. Follow the proof of Theorem 3. by Fisher and Sessa to line 6 on page 48. We then have: $Sx_{2n} \to z$, $Jx_{2n+1} \to z$, $Tx_{2n-1} \to z$, and $Ix_{2n} \to z$.

Suppose that I is continuous. Then $ISx_n \to Iz$, and $IIx_{2n} \to Iz$. But $SIx_{2n} \to Iz$ also, by Proposition (1.1) 2.(a), since I and S are compatible. Then as in [25], line 10, page 48, to line 5, page 49, we obtain $Iz = z$ and $Sz = z$.

Since $S(X) \subseteq J(X)$, $\exists\, z'$ such that $Jz' = z$. As in [25], line 9, page 49, to line 12, page 49, we have $Tz' = z$. But $Jz' = Tz'$ implies that T and J commute at z', by Proposition (1.1)1. This implies $Tz = TJz' = JTz' = Jz$. That $Tz = Jz = z$ follows from the last five lines of page 49, [24]. Therefore, I, S, T, and J have a common fixed point z if I is continuous.

The proof for the case in which J is continuous is analogous to the preceding proof. In fact, the remainder of the proof in [25] beginning with line 6, page 50, holds if the phrase, " Since __ and __ are compatible" is substituted for every appearance of " Since __ and __ weakly commute", with one exception. Beginning with the fifth line from the bottom of page 51, we would say, " Since S and I are compatible, the fact that $Sz'' = z = Iz''$ implies $Iz = ISz'' = SIz'' = Sz$. We thus have $Iz = Sz$ and $z = Tz = Jz$ from above. But then,

$$\begin{aligned} d(Sz, z\,) = d(Sz, Tz) &< g(\, d(Iz, Jz), d(Iz, Sz), d(Jz, Tz)\,) \\ &= g(\, d(Sz, z), 0, 0\,) < h\, d(Sz, z), \end{aligned}$$

and this implies that $Sz = z$. Thus, z is a common fixed point of I, J, S, and T." □

The following theorem generalizes Theorem 3.1 of M. S. Kahn and M. Swaleh in [26]. The only change in the statement of theorem is to require $\{A,S\}$ and $\{A,T\}$ to be <u>compatible</u> pairs as opposed to <u>weakly</u> <u>commuting</u> pairs.

THEOREM 2.8. Let A, S, and T be self maps of a complete metric space (X, d). Furthermore, suppose that

(a) $d(Sx, Ty) \leq a_1 d(Sx, Ax) + a_2 d(Ty, Ay) + a_3 d(Sx, Ay) + a_4 d(Ty, Ax) + a_5 d(Ax, Ay)$ for $x,y \in X$, where each $a_i \geq 0$ and $\max\{\, a_2 + a_4,\ a_3 + a_4 + a_5\,\} < 1$,

(b) A is continuous,

(c) $\{A, S\}$ and $\{A,T\}$ are compatible pairs, and

(d) $\exists$ a sequence which is asymptotically $S-$regular as well as $T-$regular with respect to A.

Then A, S, and T have a unique common fixed point.

PROOF. The proof is the same as the proof of Theorem 3.1 in [26] down to ten lines from the bottom of page 986. Now since $Ax_n \to z$ and $Sx_n \to z$, $A^2x_n \to Az$ and $ASx_n \to Az$ since A is continuous. But then (Proposition (1.1)), $SAx_n \to Az$ since $\{A, S\}$ is a compatible pair. Similarly, we conclude that $ATx_n \to Az$ and $TAx_n \to Az$. The remainder of the proof is as in [26] . □

We now consider compatibility and/or generalizations thereof in the context of multi-valued maps.

3. MULTI-VALUED FUNCTIONS AND COMPATIBILITY.

We shall consider three papers involving multi-valued functions. The first two let B(X) denote the set of bounded subsets of a complete metric space (X,d) and define a function δ: $B(X)xB(X) \to [0,\infty)$ by $\delta(A,B) = \sup \{ d(a, b): a \in A \text{ and } b \in B \}$. See [27] or [28] for a discussion and listing of properties of δ. We do note that $0 \leq \delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for $A,B,C \in B(X)$, and $\delta(A, B)=0$ iff $A=B=\{a\}$. If $x \in X$, we write $\delta(x, A)$ for $\delta(\{x\}, A)$ when convenient and confusion is not likely.

If $\{A_n\}$ is a sequence in B(X), we say that $\{A_n\}$ converges to $A \subseteq X$, and write $A_n \to A$, iff

(i) $a \in A$ implies that $a = \lim_{x\to\infty} a_n$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $n \in N$, and

(ii) for any $\epsilon>0$ $\exists$ $m \in N$ such that $A_n \subseteq A_\epsilon = \{x \in X: d(x, a) < \epsilon \text{ for some } a \in A \}$ for $n > m$.

We need the following lemmas.

LEMMA 3.1 ([27]) Suppose $\{A_n\}$ and $\{B_n\}$ are sequences in B(X) and (X, d) is a complete metric space. If $A_n \to A \in B(X)$ and $B_n \to B \in B(X)$, then $\delta(A_n, B_n) \to \delta(A, B)$.

LEMMA 3.2 ([28]) If $\{A_n\}$ is a sequence of nonempty bounded sets in the complete metric space (X,d) and if $\lim_{n\to\infty} \delta(A_n,\{y\}) = 0$ for some $y \in X$, then $A_n \to \{y\}$.

To define "compatibility" in this context, we say the following.

DEFINITION 3.1. Let (X, d) be a metric space. Let $I : X \to X$ and $F: X \to B(X)$. F and I are δ-compatible iff $IFx \in B(X)$ for $x \in X$ and $\delta(IFx_n, FIx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Ix_n \to t$ and $Fx_n \to \{t\}$ for some $t \in X$.

Observe that even though the conditions of the above definition are satisfied non-vacuously, F need not be single valued. Consider, e.g., $I:\mathbb{R}\to\mathbb{R}$ and $F: \mathbb{R} \to B(\mathbb{R})$ defined by $Ix = x/3$ and $Fx = [0, x/2]$, where $\mathbb{R}$ denotes the reals with the usual topology.

The following result regarding δ-compatibility will prove useful. Note that by definition, a function $F:X\to B(X)$ is *continuous* iff $x_n \to z$ in (X,d) implies $Fx_n \to Fz$ in B(X).

PROPOSITION 3.1. Let (X,d) be a complete metric space. Suppose $I:X\to X$, $F:X\to B(X)$, and I and F are δ-compatible.

(i) Suppose the sequence $\{Fx_n\}$ converges to $\{z\}$ and $\{Ix_n\}$ converges to z. If I is continuous,

then $FIx_n \to \{Iz\}$.

(ii) If $\{Iu\} = Fu$ for some $u \in X$, then $FIu = IFu$.

PROOF. We first prove (i). Suppose that I is continuous. Since $Fx_n \to \{z\}$, $IFx_n \to \{Iz\}$ by the definition of convergence of sets, and therefore $\delta(IFx_n, \{Iz\}) \to \delta(\{Iz\},\{Iz\}) = 0$ (Lemma 3.1). But

$$\delta(FIx_n, \{Iz\}) \leq \delta(FIx_n, IFx_n) + \delta(IFx_n, \{Iz\}) \text{ , for } n \in N.$$

Since it is also true that $Ix_n \to z$ and the pair $\{I, F\}$ is δ-compatible, $\delta(FIx_n, IFx_n) \to 0$ as $n \to \infty$. Consequently, the above implies that $\delta(FIx_n, \{Iz\}) \to 0$. Therefore, $FIx_n \to \{Iz\}$ by Lemma 3.2.

To see that (ii) also holds, let $x_n = u$ for $n \in N$. Then $Ix_n \to Iu$ and $Fx_n \to Fu = \{Iu\}$, so that $\delta(FIu, IFu) = \delta(FIx_n, IFx_n) \to 0$ by δ-compatibility; i.e., $IFu = FIu$, a singleton. □

We now state and prove the first theorem, which extends Theorem 1. of Fisher in [27] by replacing commutativity of maps $I:X \to X$ and $F:X \to B(X)$ by δ-compatibility. Note that in the following we use $\cup F(X)$ to denote $\{ y \in X: y \in F(x) \text{ for some } x \in X. \}$.

THEOREM 3.1. Let I and J be selfmaps of a complete metric space (X,d), and let $F,G:X \to B(X)$. Suppose $\exists\, c \in (0,1)$ such that for all $x,y \in X$:

$$\delta(Fx, Gy) \leq c \max \{ d(Ix, Jy), \delta(Ix, Gy), \delta(Jy, Fx) \}. \tag{3.1}$$

Suppose the mappings F and I are δ-compatible and G and J are δ-compatible, that $\cup F(X) \subseteq I(X)$ and $\cup G(X) \subseteq J(X)$. If F or I and G or J are continuous, then F, G, I, and J have a unique common fixed point. Moreover, $Fz = Gz = \{z\}$ is the unique common fixed points of F and I and of G and J.

PROOF. Follow the proof of Theorem 1. by Fisher ([27]) from page 16 to line 4 page 18. Note that we have $Ix_n, Jy_n \to z \in X$ and $Fx_n, Gy_n \to \{z\}$.

Now suppose that I is continuous. Then $IIx_n \to Iz$. But I and F are δ-compatible and I is continuous; therefore, $FIx_n \to \{Iz\}$ by Proposition 3.1 (i). Consequently, since (3.1) yields

$$\delta(FIx_n, Gy_n) \leq c \max \{ d(IIx_n, Jy_n), \delta(IIx_n, Gy_n), \delta(Jy_n, FIx_n) \}$$

for $n \in N$, as $n \to \infty$ we obtain $\delta(Iz, z) \leq c\, \delta(Iz, z)$ by Lemma 3.1. Thus $Iz = z$. Then follow Fisher to obtain, $z = Jz = Iz$, and $Fz = Gz = \{z\}$.

Next suppose that F is continuous. Then $FIx_n \to Fz$ since $Ix_n \to z$. And by construction, $Ix_n \in Fx_{n-1}$, so $IIx_n \in IFx_{n-1}$ for all n. The inequality (3.1) thus implies:

$$\begin{aligned}\delta(FIx_n, Gy_n) &\leq c \max \{ d(IIx_n, Jy_n), \delta(IIx_n, Gy_n), \delta(Jy_n, FIx_n) \} \\ &\leq c \max \{ \delta(IFx_{n-1}, Jy_n), \delta(IFx_{n-1}, Gy_n), \delta(Jy_n, FIx_n) \} \\ &\leq c \max \{ \delta(FIx_{n-1}, Jy_n) + \delta_n, \delta(FIx_{n-1}, Gy_n) + \delta_n, \delta(Jy_n, FIx_n) \},\end{aligned}$$

for $n \in N$, where $\delta_n = \delta(IFx_n, FIx_n) \to 0$ as $n \to \infty$ by compatibility. Thus $\delta(Fz, \{z\}) \leq c\, \delta(Fz, \{z\})$; i.e., $Fz = \{z\}$.

Now follow Fisher ([27]) to the sixth line from the bottom of page 18. We have a point u such that $Iu = z$ and $Fu = \{z\}$. Since I and F are δ-compatible, $IFu = FIu$ by

Proposition 3.1(ii). Thus, $\{z\} = Fz = FIu = IFu = \{Iz\}$. The remainder of the proof follows as in [27]. □

The other paper we consider and which utilizes the function δ: $B(X)xB(X) \to [0, \infty)$ for which the above definitions and lemmas pertain, is the paper [28] by Imdad, Kahn, and Sessa. (X,d) is assumed to be a complete metric space and $I: X \to X$, $F: X \to B(X)$.

The authors introduce a generalized commutativity by saying that F and I *slightly commute* iff $IFx \in B(X)$ and $\delta(FIx, IFx) \leq \max\{\delta(Ix, Fx), \delta(Fx, Fx)\}$ for $x \in X$. If F is single-valued, the inequality reduces to $d(FIx, IFx) \leq d(Ix, Fx)$ for $x \in X$, so that F and I are weakly commuting. As noted in the introduction, weakly commuting pairs are compatible, but the converse need not hold. And it is clear that if F and I slightly commute, then F and I are δ-compatible; thus δ-compatibility does generalize slight commutativity.

We generalize Theorem 5. in [28] by substituting δ-compatibility for slight commutativity. Note that $\psi:[0,\infty) \to [0, \infty)$, is nondecreasing, right continuous, and satisfies $\psi(t)<t$ for $t>0$.

THEOREM 3.2. Let the maps $F:X \to B(X)$ and $I: X \to X$ satisfy for $x,y \in X$:

$$\delta(Fx, Fy) \leq \psi(\max\{d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Fy), \delta(Ix, Fy), \delta(Iy, Fx)\}).$$

If there exists $x_0 \in X$ such that $\sup\{\delta(Fx_n, Fx_1) : n=0,1,2,\ldots\} < +\infty$, if F and I are δ-compatible, if $\cup F(X) \subseteq I(X)$, and if F or I is continuous, then F and I have a unique common fixed point z; furthermore, $Fz = \{z\}$.

PROOF. Begin as in the proof of Theorem 5 in [28]. Then replace the second paragraph of the proof (page 294) by the following. "As in [2], we have $Ix_n \to z \in X$ and $Fx_n \to \{z\}$. Consequently, compatibility implies that $\delta(FIx_n, IFx_n) \to 0$ as $n \to \infty$", their property (4.2).

Then continue as in [28] until lines 1 and 2 of page 295, which we replace by the following observation "Since $\{Iw\}=\{z\}=Fw$, $F(\{z\}) = FIw = IFw = \{Iz\}$, by Proposition 3.1(ii) and compatibility. Thus $\{z\}=Fz=\{Iz\}$."

The rest of the proof is as in [28]. □

The third and final paper involving multi-valued functions is the paper [29] by Singh, Ha and Cho. The authors consider multi-valued functions $S:X \to CL(X)$, the family of closed subsets of X, where (X, d) is a metric space. They utilize the "generalized Hausdorff metric", H, on CL(X). We refer the reader to [29] for the definition of this and other relatively standard concepts used, except to note that the functions $f:X \to X$ and $S:X \to CL(X)$ are said to *commute weakly* at z iff $H(fSz, Sfz) \leq D(fz, Sz)$. If f and S commute weakly at each point of X, then they commute weakly on X. Of course, $D(a,B) = \inf\{d(a,b): b \in B\}$, for $a \in X$ and $B \subseteq X$. Observe that the definition of H and weak commutativity imply that $fSx \in CL(X)$ for $x \in X$.

In this context we shall give the following "compatibility" definition.

DEFINITION 3.2. Mappings $f:X \to X$ and $S:X \to Cl(X)$ are "H − compatible" iff $fSx \in CL(X)$ for $x \in X$ and $H(Sfx_n, fSx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Sx_n \to M$ in CL(X) and $fx_n \to t \in M$.

The Definition 3.2 is basically the definition of compatible maps $S:X \to CB(X)$ and $f:X \to X$ given in [29] in the context of closed and <u>bounded</u> subsets of X. Therein, Sessa and Kaneko prove a lemma which is valid for CL(X), and which we find useful.

LEMMA 3.3. [29] Let $S: X \to CL(X)$ and $f:X \to X$ be compatible. If $fw \in Sw$, then $fSw=Sfw$.

We now state a variation of the main theorem in [29] obtained by replacing "weakly commuting" at a point by " H – compatible". The statement refers to a family F, which is the family of mappings $\phi:[0,\infty)\to[0,\infty)$ which are upper-semicontiuous and nondecreasing.

THEOREM 3.3. Let S and T be multi – valued mappings from a metric space (X,d) into CL(X). If $\exists$ a mapping $f:X\to X$ such that $S(X)\cup T(X)\subseteq f(X)$, and for each $x,y\in X$ and $\phi\in F$

$H(Sx,Ty) \leq \phi(\max\{D(fx, Sx), D(fy, Ty), D(fx, Ty), D(fy, Sx), d(fx, fy)\})$,

$\phi(t)\leq qt$ for all $t>0$ and some fixed $q\in(0,1)$,

$\exists\ x_0\in X$ such that the pair (S,T) is asymptotically regular at x_0,

and if f(X) is $(S,T; f, x_0)$ – orbitally complete,

then f, S, and T have a coincidence point. Furthermore, if z is a coincidence point of f, S, and T, and fz is a fixed point of f, then (a) fz is also a fixed point of S (resp. T) provided f and S (resp. T) are H – compatible, and (b) fz is a common fixed point of S and T provided the pairs {f,S} and {f,T} are H – compatible.

PROOF. The proof is the same as in [29], except substitute "H-compatible" for "commutes weakly at z" in lines 8 and 10, page 253 of [29].

4. RETROSPECT.

The preceding may suggest to the reader that any metric space fixed point theorem for commuting mappings obtained by using "contractive conditions" can be generalized by substituting the compatibility requirement for commutativity. The papers [30, 31, 32, 33] contain results for which this is not the case. In particular, the papers [32] and [33] by Fisher, provide examples which happen to be weakly commuting and therefore compatible for which the featured theorems are false. The question as to how far we can go in substituting compatibility for commutativity in the context of compact metric spaces is commented on in [13].

Added in Proof: The theorem of B.K. Sharma and N.K. Sahu [Common fixed points of three continuous mappings, Math. Student 59 (1991) 77-80] can also be extended to compatible mappings.

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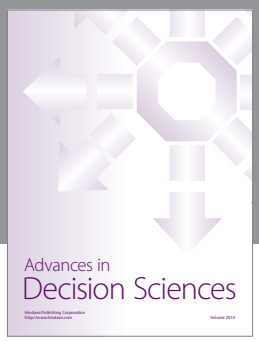

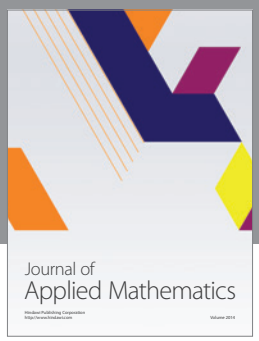

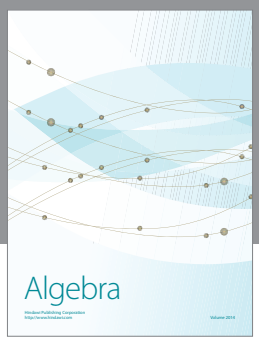

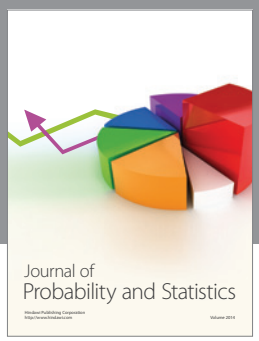

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