ON CERTAIN CLASSES OF $p$-VALENT ANALYTIC FUNCTIONS

NAK EUN CHO

Department of Applied Mathematics
National Fisheries University of Pusan
Pusan 608-737
KOREA

(Received November 5, 1991)

ABSTRACT. The objective of the present paper is to introduce a certain general class $P(p,a,\beta)(p \in \mathbb{N} \{1,2,3,...\}, 0 \leq a < p$ and $\beta \geq 0)$ of $p$-valent analytic functions in the open unit disk $\mathbb{U}$ and we prove that if $f \in P(p,a,\beta)$ then $J_{p,c}(f)$, defined by

$$J_{p,c}(f) = \frac{c + p}{c^{p}} \int_{0}^{1} t^{c-1} f(t) \, dt \quad (c \in \mathbb{N}),$$

belongs to $P(p,a,\beta)$. We also investigate inclusion properties of the class $P(p,a,\beta)$. Furthermore, we examine some properties for a class $T_{p}(a,\beta)$ of analytic functions with negative coefficients.

KEY WORDS AND PHRASES. $p$-valent analytic function, Hadamard product, integral operator, multiplier transformation, $p$-valently convex of order $\delta$.

1991 AMS SUBJECT CLASSIFICATION CODE. Primary 30C45.

1. INTRODUCTION.

Let $A_{p}$ denote the class of functions of the form

$$f(z) = z^{p} + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} \{1,2,3,...\})$$

which are analytic in the unit disk $\mathbb{U} = \{z : |z| < 1\}$. We also denote by $S_{p}$ the subclass of $A_{p}$ consisting of functions which are $p$-valent in $\mathbb{U}$.

A function $f \in A_{p}$ is said to be in the class $P(p,a)$ $(0 \leq a < p)$ if and only if it satisfies the inequality

$$\Re \left( \frac{f'(z)}{z^{p-1}} \right) > a \quad (0 \leq a < p, z \in \mathbb{U}).$$

The classes $P(1,0)$ and $P(p,0)$ were investigated by MacGregor [7] and Umezawa [11], respectively. In fact, the class $P(p,a)$ is a subclass of the class $S_{p}$ [11].

Let $f$ and $g$ be in the class $A_{p}$, with $f(z)$ given by (1.1), and $g(z)$ defined by

$$g(z) = z^{p} + \sum_{n=1}^{\infty} \delta_{n+p} z^{n+p}.$$

The convolution or Hadamard product of $f$ and $g$ is defined by
For a function $f \in A_p$ given by (1.1), Reddy and Padmanabhan [10] defined the integral operator $J_{p,c}$ $(p,c \in N)$ by

$$J_{p,c}(f) = \frac{c+p}{z^p} \int_0^z t^{-1} f(t) \, dt$$

$$= z^p + \sum_{n=1}^{\infty} \frac{c+p}{c+n+p} a_n + p z^{n+p}. \quad (1.5)$$

The operator $J_{1,c}$ was introduced by Bernardi [2]. In particular, the operator $J_{1,1}$ were studied by Libera [5] and Livingston [6].

Clearly, (1.5) yields

$$f \in A_p \Rightarrow J_{p,c} \in A_p \quad (1.6)$$

Thus, by applying the operator $J_{p,c}$ successively, we can obtain

$$J_{p,c}^n(f) = \begin{cases} 
J_{p,c}(J_{p,c}^{n-1}(f)) & (n \in N), \\
-f & (n = 0).
\end{cases} \quad (1.7)$$

We now recall the following definition of a multiplier transformation (or fractional integral and fractional derivative).

**DEFINITION** ([3]). Let the function

$$\phi(z) = \sum_{n=0}^{\infty} c_n + p z^{n+p}$$

be analytic in $U$ and let $\lambda$ be a real number. Then the multiplier transformation $I^\lambda\phi$ is defined by

$$I^\lambda\phi(z) = \sum_{n=0}^{\infty} (n + p + 1)^{\lambda} c_n + p z^{n+p} \quad (z \in U). \quad (1.9)$$

The function $I^\lambda\phi$ is clearly analytic in $U$. It may be regarded as a fractional integral (for $\lambda > 0$) or fractional derivative (for $\lambda < 0$) of $\phi$. Furthermore, in terms of the Gamma function, we have

$$I^\lambda\phi(z) = \frac{1}{\Gamma(z)} \int_0^1 (\log t)^{\lambda-1} \phi(zt) \, dt \quad (\lambda > 0). \quad (1.10)$$

**DEFINITION** 2. The fractional derivative $D^\lambda\phi$ of order $\lambda \geq 0$, for an analytic function $\phi$ given by (1.8), is defined by

$$D^\lambda\phi(z) = I^{-\lambda}\phi(z) = \sum_{n=0}^{\infty} (n + p + 1)^{\lambda} c_n + p z^{n+p} \quad (\lambda \geq 0, z \in U). \quad (1.11)$$

Making use of Definition 2, we now introduce an interesting generalization of the class $P(p,\alpha)$ of functions in $A_p$ which satisfy the inequality (1.2).

**DEFINITION** 3. A function $f \in A_p$ is said to be in the class $P(p,\alpha,\beta)$ if and only if

$$(p+1)^{-\beta} D^\beta f \in P(p,\alpha) \quad (0 \leq \alpha < p, \beta \geq 0)$$

Observe that $P(p,\alpha,0) = P(p,\alpha)$. Furthermore, since $f \in A_p$, it follows from (1.1) and (1.9) that
In particular, the class $P(1, \alpha, \beta)$ was introduced by Kim, Lee, and Srivastava [4].

2. SOME INCLUSION PROPERTIES.

In our present investigation of the general class $P(p, \alpha, \beta)$ $(0 \leq \alpha < p, \beta \geq 0)$, we need the following lemma.

**LEMMA 2.1([1]).** Let $M(z)$ and $N(z)$ be analytic in $U$, $N(z)$ map $U$ onto a many sheeted starlike region of order $\gamma$ $(0 \leq \gamma < p)$ and

$$M(0) = N(0) = 0, \quad \frac{M'(0)}{N'(0)} = p, \quad \Re \left( \frac{M'(z)}{N'(z)} \right) > \gamma.$$ 

Then we have

$$\Re \left( \frac{M(z)}{N(z)} \right) > \gamma \quad (0 \leq \gamma < p, p \geq 1).$$

By using Lemma 2.1, we can prove

**THEOREM 2.1.** Let the function $f(z)$ be in the class $P(p, \alpha, \beta)$. Then $J_{p,c}(f)$ defined by (1.5) is also in the class $P(p, \alpha, \beta)$.

**PROOF.** A simple calculation shows that

$$\frac{d}{dz} \left( J_{p,c}(f) \right) = \frac{c + p}{z^p + p} \int_0^z \frac{d}{dt} \left( D^\beta f(t) \right) dt \quad (2.1)$$

where the operators $J_{p,c}$ $(c \in \mathbb{N})$ and $D^\lambda$ $(\lambda \geq 0)$ are defined by (1.5) and (1.11), respectively. In view of (2.1), we get

$$M(z) = \frac{c + p}{(p + 1)^p} \int_0^z t^c \frac{d}{dt} \left( D^\beta f(t) \right) dt \quad \text{and} \quad N(z) = z^p + c, \quad (2.2)$$

so that

$$\Re \left( \frac{M'(z)}{N'(z)} \right) = \Re \left( \frac{(p + 1)^{-\beta} \frac{d}{dz} D^\beta f(z)}{z^{p-1}} \right). \quad (2.3)$$

Since, by hypothesis, $f \in P(p, \alpha, \beta)$, the second member of (2.3) is greater than $\alpha$, and hence

$$\Re \left( \frac{M'(z)}{N'(z)} \right) > \alpha \quad (0 \leq \alpha < p). \quad (2.4)$$

Thus, by Lemma 2.1, we have

$$\Re \left( \frac{M(z)}{N(z)} \right) = \Re \left( \frac{(p + 1)^{-\beta} \frac{d}{dz} J_{p,c}(f)}{z^{p-1}} \right) > \alpha \quad (0 \leq \alpha < p, \beta \geq 0), \quad (2.5)$$

which completes the proof of Theorem 2.1.

Let $f \in A_p$ be given by (1.1). Suppose also that
\[ F_m(f) = J_{p,c_1}\left(\cdots \left( J_{p,c_m}(f)\right)\right) \]
\[ = z^p + \sum_{n=1}^{\infty} \frac{(c_1 + p)\cdots (c_m + p)}{(c_1 + p + n)\cdots (c_m + p + n)} \frac{a_{n+p}}{z^{n+p}} \quad (c_j \in \mathbb{N}(j = 1, \ldots, m), m \in \mathbb{N}). \]  

(2.6)

Then, by Theorem 2.1, we have

**COROLLARY 2.1.** Let the function \( f(z) \) be in the class \( P(p,\alpha,\beta) \). Then the function \( F_m(f) \) defined by (2.6) is also in the class \( P(p,\alpha,\beta) \).

The next inclusion property of the class \( P(p,\alpha,\beta) \), contained in Theorem 2.2 below, would involve the operator \( J_{p,1}^\lambda(\lambda > 0) \) defined by

\[ J_{p,1}^\lambda(f) = (1 + p)^{\lambda} I_{1}^{\lambda} f(z) \]  
(\( \lambda > 0, f \in A_p \)).

(2.7)

For \( \lambda = m \in \mathbb{N} \), we have

\[ J_{p,1}^m(f) = (1 + p)^{m} I_{1}^{m} f(z) \]
\[ = \frac{(1 + p)^{m}}{(m - 1)!} \int_0^1 (\log t)^{m-1} f(t) dt. \]

(2.8)

Clearly, we have

\[ f \in A_p \Rightarrow J_{p,1}^\lambda(f) \in A_p \]  
(\( \lambda > 0 \)).

(2.9)

**THEOREM 2.2.** Let the function \( f(z) \) be in the class \( P(p,\alpha,\beta) \). Then the function \( J_{p,1}^\lambda(\lambda > 0) \) defined by (2.7) is also in the class \( P(p,\alpha,\beta) \).

**PROOF.** Making use of (1.9) and (1.11), the definition (2.7) yields

\[ (p + 1)^{-\beta} D^\beta (J_{p,1}^\lambda(f)) = J_{p,1}^\lambda((p + 1)^{-\beta} D^\beta f) \]
(\( \beta \geq 0, \lambda > 0, f \in A_p \)).

(2.10)

Therefore, setting

\[ g(z) = (p + 1)^{-\beta} D^\beta f \]  
and \( G(z) = J_{p,1}^\lambda(g) \),

we must show that

\[ \text{Re}\left\{ \frac{G'(z)}{z^{\lambda - 1}} \right\} > 0 \quad (0 < \alpha < p) \]

(2.12)

whenever \( f \in P(p,\alpha,\beta) \).

From the integral representation in (1.10), we obtain

\[ G'(z) = \frac{(p + 1)^\lambda}{\Gamma(\lambda)} \int_0^1 (\log t)^{\lambda-1} g'(zt) dt \]  
(\( \lambda > 0 \)).

(2.13)

so that

\[ \text{Re}\left\{ \frac{G'(z)}{z^{\lambda - 1}} \right\} = \frac{(p + 1)^\lambda}{\Gamma(\lambda)} \int_0^1 (\log t)^{\lambda-1} t^{\lambda - 1} \text{Re}\left\{ \frac{g'(zt)}{(zt)^{p-1}} \right\} dt \]  
(\( \lambda > 0 \)).

(2.14)

Since \( f \in P(p,\alpha,\beta) \), we have

\[ \text{Re}\left\{ \frac{g'(zt)}{(zt)^{p-1}} \right\} > 0 \quad (0 < \alpha < p, 0 \leq t \leq 1). \]

(2.15)
CERTAIN CLASSES OF p-VALENT ANALYTIC FUNCTIONS 323

and hence (2.14) yields

\[ \text{Re}\left\{ \frac{G'(z)}{z^p-1} \right\} = \frac{(p+1)^\alpha}{\Gamma(\lambda)} \int_0^1 (\log \frac{1}{t+1})^{\lambda-1} t e^{-\lambda-1} t dt = \alpha \]

(0 ≤ α < p, λ > 0), \hspace{1cm} (2.16)

which completes the proof of Theorem 2.2.

COROLLARY 2.2. If 0 ≤ α < p and 0 ≤ β < γ, then \( P(p, \alpha, \gamma) \subset P(p, \alpha, \beta) \).

PROOF. Setting \( \lambda = \gamma - \beta > 0 \) in Theorem 2.2, we observe that

\[ f \in P(p, \alpha, \gamma) \Rightarrow J_{p,1}^{-\beta}(f) \in P(p, \alpha, \gamma) \]

\[ \Leftrightarrow (p+1)^{-\beta} D^\gamma (J_{p,1}^{-\beta}(f)) \in P(p, \alpha) \]

\[ \Leftrightarrow (p+1)^{-\beta} D^\beta f \in P(p, \alpha) \]

\[ \Leftrightarrow f \in P(p, \alpha, \beta), \]

and the proof of Corollary 2.2 is completed.

Next we define a function \( h \in A_p \) by

\[ h(z) = z^p + \sum_{n=1}^{\infty} \left( \frac{n+p+1}{p+1} \right) z^n \]

(\( z \in U \)). \hspace{1cm} (2.18)

Then, in terms of the Hadamard product defined by (1.4), we have

\[ (h \ast f)(z) = \frac{1}{p+1} \{ f(z) + zf'(z) \} \]

which, when compared with (1.11) with \( m = 1 \), yields

\[ (h \ast f)(z) = \frac{1}{p+1} D^1 f. \]

We now need the following lemma for another inclusion property of the class \( P(p, \alpha, \beta) \).

LEMMA 2.2([8]). Let \( \varphi(u, v) \) be a complex valued function such that

\( \varphi: D \rightarrow \mathbb{C}, \)

(\( D \subset \mathbb{C} \times \mathbb{C} (C \text{ is the complex plane}) \)),

and let \( u = u_1 + iu_2, \ v = v_1 + iv_2 \). Suppose that the function \( \varphi(u, v) \) satisfies

(i) \( \varphi(u, v) \) is continuous in \( D \),

(ii) \( (1,0) \in D \) and \( \text{Re}\{\varphi(1,0)\} > 0 \),

(iii) for all \( (iu_2, v_1) \in D \) such that \( v_1 \leq -\frac{1+u_2^2}{2}, \quad \text{Re}\{\varphi(iu_2, v_1)\} \leq 0 \).

Let \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) be analytic in the unit disk \( U \) such that \( (p(z), zp'(z)) \in D \) for all \( z \in U \). If

\[ \text{Re}\{\{p(z), zp'(z)\}\} > 0 \quad (z \in U), \]

then \( \text{Re}\{p(z)\} > 0 (z \in U) \).

THEOREM 2.3. If 0 ≤ α < p and β ≥ 0, then

\[ P(p, \alpha, \beta + 1) \subset P(p, \mu, \beta) \quad (\mu = \frac{2\alpha(p+1)+p}{2(p+1)+1}). \]

PROOF. Let the function

\[ F(z) = \frac{1}{p+1} \{ f(z) + zf'(z) \} \]

(\( f \in A_p \)). \hspace{1cm} (2.22)

First, we shall show that
\[
\text{Re}\left\{ \frac{f'(z)}{z^{p-1}} \right\} > \frac{2\alpha(p+1) + p}{2(p+1) + 1} \quad (0 \leq \alpha < p, z \in U), \tag{2.23}
\]

whenever
\[
\text{Re}\left\{ \frac{F'(z)}{z^{p-1}} \right\} > \alpha \quad (0 \leq \alpha < p, z \in U). \tag{2.24}
\]

By the differentiation of \( F(z) \), we obtain
\[
F'(z) = \frac{1}{p+1} [2f'(z) + zf''(z)]. \tag{2.25}
\]

We define the function \( p(z) \) by
\[
F'(z) = \frac{f'(z)}{p^{p-1}} = \gamma + (1 - \gamma)p(z) \tag{2.26}
\]

with \( \gamma = \frac{2\alpha(p+1) + p}{2(p+1) + 1} \) (0 \leq \gamma < 1). Then \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) is analytic in \( U \). By using (2.25) and (2.26), we obtain
\[
\text{Re}\left\{ \frac{F'(z)}{z^{p-1}} \right\} > \alpha \quad (0 \leq \alpha < p), \tag{2.27}
\]

Hence, in view of \( \text{Re}\left\{ \frac{F'(z)}{z^{p-1}} \right\} > \alpha \), we have
\[
\text{Re}\left\{ \phi(p(z), zp'(z)) \right\} > 0, \tag{2.28}
\]

where \( \phi(u,v) \) is defined by
\[
\phi(u,v) = \frac{1}{p+1} ((p^2 + p)(\gamma + (1 - \gamma)u) + p(1 - \gamma)v) - \alpha \tag{2.29}
\]

with \( u = u_1 + iu_2 \) and \( v = v_1 + iv_2 \). Then we see that
(i) \( \phi(u,v) \) is continuous in \( D = C \times C \),
(ii) \( (1,0) \in D \) and \( \text{Re}\{\phi(1,0)\} = p - \alpha > 0 \),
(iii) for all \((iu_2,v_1)\in D\) such that \( v_1 \leq -\frac{(1 + u_2^2)}{2} \),
\[
\text{Re}\{\phi(iu_2,v_1)\} = \frac{1}{p+1} ((p^2 + p)\gamma + p(1 - \gamma)v) - \alpha \\
\begin{align*}
&\leq \frac{1}{p+1} \left( (p^2 + p)\gamma - p(1 - \gamma)\frac{(1 + u_2^2)}{2} \right) - \alpha \leq 0
\end{align*}
\]

for \( \gamma = \frac{2\alpha(p+1) + p}{2(p+1) + 1} \). Consequently, \( \phi(u,v) \) satisfies the conditions in Lemma 2.2. Therefore, we have
\[
\text{Re}\left\{ \frac{f'(z)}{z^{p-1}} \right\} > p\gamma = \frac{2\alpha(p+1) + p}{2(p+1) + 1}. \tag{2.30}
\]

Next, in view of (2.20) and above arguments, we have
\[
f \in P(p,\alpha,\beta+1) \Leftrightarrow (p+1)^{-\beta}D^{\beta+1}f \in P(p,\alpha) \\
\Rightarrow h \ast ((p+1)^{-\beta}D^{\beta+1}f) \in P(p,\alpha) \\
\Rightarrow (p+1)^{-\beta}D^{\beta+1}f \in P(p,\mu) \quad (\mu = \frac{2\alpha(p+1) + p}{2(p+1) + 1}) \\
\Leftrightarrow f \in P(p,\mu,\beta), \tag{2.31}
\]

which evidently proves Theorem 2.3.
REMARK. Since $0 \leq \alpha < p$, we have

$$\mu = \frac{2\alpha(p+1) + p}{2(p+1)} > \alpha,$$

and hence $P(p, \mu, \beta) \subset P(p, \alpha, \beta)$.

3. THE CONVERSE PROBLEM.

Let $T_p$ denote the class of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}, a_{n+p} \geq 0)$$

which are analytic in $U$ and let $T_p(a, \beta) = T_p \cap P(p, a, \beta)$.

In this section, we investigate the converse problem of integrals defined by (1.5) for the class $T_p(a, \beta)$.

LEMMA 3.1. Let $f \in T_p$. Then $f \in T_p(a, \beta)$ if and only if

$$\sum_{n=1}^{\infty} (n+p)^{(n+p+1)} a_{n+p} \leq p - \alpha. \quad (3.1)$$

PROOF. Suppose that

$$\sum_{n=1}^{\infty} (n+p)^{(n+p+1)} a_{n+p} \leq p - \alpha.$$

It is sufficient to show that the values for $\frac{(p+1)^{-\beta}(D^\beta f)'_z^{p-1}}{z^p}$ lie in a circle centered at $p$ whose radius is $p - \alpha$. Indeed, we have

$$\left| \frac{(p+1)^{-\beta}(D^\beta f)'_z^{p-1}}{z^p} - p \right| = \left| - \sum_{n=1}^{\infty} (n+p)^{(n+p+1)} a_{n+p} z^n \right|$$

$$\leq \sum_{n=1}^{\infty} (n+p)^{(n+p+1)} a_{n+p} |z|^n$$

$$< \sum_{n=1}^{\infty} (n+p)^{(n+p+1)} a_{n+p} \leq p - \alpha. \quad (3.2)$$

Conversely, assume that

$$Re\left\{ \frac{(p+1)^{-\beta}(D^\beta f)'_z^{p-1}}{z^p} \right\} > \alpha (0 \leq \alpha < p) \quad (3.3)$$

which is equivalent to

$$Re\left\{ \sum_{n=1}^{\infty} (n+p)^{(n+p+1)} a_{n+p} z^n \right\} < p - \alpha. \quad (3.4)$$

Choose values of $z$ on the real axis so that

$$\sum_{n=1}^{\infty} (n+p)^{(n+p+1)} a_{n+p} z^n$$

is real. Letting $z \to 1$ along the real axis, we obtain

$$\sum_{n=1}^{\infty} (n+p)^{(n+p+1)} a_{n+p} \leq p - \alpha.$$

The proof is completed.
THEOREM 3.1. Let $F \in T_p(\alpha, \beta)$ and $f(z) = \left[\frac{1-c}{p+c}\right] [z^c F(z)]^\delta$ $(c \in \mathbb{N})$. Then the function $f(z)$ belongs to the class $T_p(\delta, \beta)$ $(0 \leq \delta < p)$ for $|z| < r$, where

$$r = \inf_{n \geq 1} \left[\frac{(p-\delta)(p+c)}{(p-\alpha)(n+p+c)}\right]^\frac{1}{n}.$$  

(3.5) The result is sharp.

PROOF. Let $F(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^n + p$. Then it follows from (1.5) that

$$f(z) = \frac{1-c}{p+c} \frac{d}{dz} [z^c F(z)] = z^p - \sum_{n=1}^{\infty} \left(\frac{n+p+c}{p+c}\right) a_{n+p} z^n.$$  

(3.6) To prove the result, it suffices to show that

$$\left|\frac{(p+1)^{-\beta}(D^2f)'}{z^{p-1}} - p\right| \leq p-\delta$$  

(3.7) for $|z| \leq r$. Now

$$\left|\frac{(p+1)^{-\beta}(D^2f)'}{z^{p-1}} - p\right| = - \sum_{n=1}^{\infty} (n+p+1)^\delta \left(\frac{n+p+c}{p+c}\right) a_{n+p} |z|^n.$$  

(3.8) Thus we have

$$\left|\frac{(p+1)^{-\beta}(D^2f)'}{z^{p-1}} - p\right| \leq p-\delta$$  

(3.9) if

$$\sum_{n=1}^{\infty} (n+p+1)^\delta \left(\frac{n+p+c}{p+c}\right) a_{n+p} |z|^n \leq p-\delta.$$  

(3.10) But Lemma 3.1 confirms that

$$\sum_{n=1}^{\infty} (n+p+1)^\delta a_{n+p} \leq p-\alpha.$$  

(3.11) Therefore (3.10) will be satisfied if

$$\left(\frac{n+p}{p-\delta}\right) \left(\frac{n+p+c}{p+c}\right) |z|^n \leq \left(\frac{n+p}{p-\alpha}\right)$$  

(3.12) for each $n \in \mathbb{N}$, or if

$$|z| \leq \left[\frac{p-\delta}{p-\alpha}\left(\frac{p+c}{n+p+c}\right)^{\frac{1}{n}}.$$  

(3.13) The required result follows now from (3.13). Sharpness follows if we take

$$F(z) = z^p - \left(\frac{p-\alpha}{n+p}\right)^\delta a_{n+p} z^n + p$$  

(3.14) for each $n \in \mathbb{N}$.

THEOREM 3.2. Let $F \in T_p(\alpha, \beta)$ and $f(z) = \left[\frac{1-c}{p+c}\right] [z^c F(z)]^\delta$ $(c \in \mathbb{N})$. Then the function $f(z)$ $p$-valently convex of order $\delta$ $(0 \leq \delta < p)$ in the disk...
\[ |z| < r^* = \inf_{n \geq 1} \left[ \frac{\frac{p(p - \delta)}{(n + p + \delta)(p - \alpha)} \left( \frac{n + p + c}{p + c} \right)^{n + p + 1}}{(n + p + \delta)(p - \alpha)} \left( \frac{n + p + 1}{p + 1} \right)^{n + p + \frac{1}{n}} \right]. \] (3.15)

The result is sharp.

**PROOF.** To prove the theorem, it is sufficient to show that
\[ \left| 1 + \frac{zf''(z)}{f'(z)} \right| - p \leq p - \delta \] (3.16)
for \(|z| \leq r^*\). In view of (3.6), we have
\[ \frac{-1}{p - \sum_{n=1}^{\infty} n(n+p)(n+p+c) a_{n+p} n^{n+p} z^{n+p-1}} \leq \frac{-1}{p - \sum_{n=1}^{\infty} n(n+p)(n+p+c) a_{n+p} n^{n+p} z^{n+p-1}} \]
(3.17)
Thus
\[ \left| 1 + \frac{zf''(z)}{f'(z)} \right| - p \leq p - \delta \] (3.18)
if
\[ \frac{-1}{p - \sum_{n=1}^{\infty} n(n+p)(n+p+c) a_{n+p} n^{n+p} z^{n+p-1}} \leq p - \delta. \] (3.19)
or
\[ \sum_{n=1}^{\infty} \frac{n(n+p)(n+p+c)}{p(p - \delta)} \left( \frac{n + p + c}{p + c} \right)^{n + p + \frac{1}{n}} a_{n+p} n^{n+p} z^{n+p-1} \leq 1. \] (3.20)

But from Lemma 3.1, we obtain
\[ \sum_{n=1}^{\infty} \frac{n(n+p)(n+p+c)}{p(p - \delta)} \left( \frac{n + p + 1}{p + 1} \right)^{n + p + \frac{1}{n}} a_{n+p} \leq 1. \] (3.21)
Hence \( f(z) \) is \( p \)-valent convex of order \( \delta \) (\( 0 \leq \delta < p \)) if
\[ \frac{n(n+p)(n+p+c)}{p(p - \delta)} \left( \frac{n + p + c}{p + c} \right) |z|^{n} \leq \left( \frac{n + p}{p - \delta} \right) \left( \frac{n + p + 1}{p + 1} \right)^{\frac{1}{n}}, \] (3.22)
or
\[ |z| \leq \left[ \frac{p(p - \delta)}{(n + p + \delta)(p - \alpha)} \left( \frac{p + c}{n + p + c} \right) \left( \frac{n + p + 1}{p + 1} \right)^{\frac{1}{n}} \right]. \] (3.23)
for each \( n \in \mathbb{N} \). This completes the proof of the theorem. The result is sharp for the function given by (3.14).

**REFERENCES**


Submit your manuscripts at
http://www.hindawi.com