COMMON STATIONARY POINTS FOR SET-VALUED MAPPINGS

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ABSTRACT. Several theorems on stationary points for set-valued mappings have obtained. These are improvements upon some earlier results due to Fisher.

KEY WORDS AND PHRASES. Generalized Hausdorff distance, nearly-densifying mappings, orbit, common stationary points.

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1. INTRODUCTION AND PRELIMINARIES.
In this paper, we prove several common stationary point theorems for four set-valued mappings, which are improvements upon some earlier results obtained by Fisher [1], [2], [3].

Let \((X, d)\) be a metric space and \(CL(X)\) be the class of all nonempty closed subset of \(X\). For \(z \in X\) and \(A \subseteq X\), let \(D(z, A) = \inf \{d(z, y) : y \in A\}\).

DEFINITION 1.1. For \(A, B \in CL(X)\), define

\[
H(A, B) = \begin{cases} \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(A, y)\}, & \text{if it exists}, \\ \infty, & \text{otherwise}. \end{cases}
\]

Then \(H\) is called the \textit{generalized Hausdorff distance function} for the class \(CL(X)\) induced by the metric \(d\).

DEFINITION 1.2. For \(A, B \in CL(X)\), define \(h: CL(X) \times CL(X) \rightarrow R^+ \) by

\[
h(A, B) = \begin{cases} \sup\{d(x, y) : x \in A, y \in B\}, & \text{if it exists}, \\ \infty, & \text{otherwise}. \end{cases}
\]

DEFINITION 1.3. A set-valued mapping \(S: X \rightarrow CL(X)\) is said to be \textit{nearly-densifying} if \(\alpha(S(A)) < \alpha(A)\) for any bounded and \(S\)-invariant subset of \(X\) with \(\alpha(A) > 0\), where \(\alpha\) is the Kuratowski's measure of non-compactness.

DEFINITION 1.4. Let \(F, G, S, T : X \rightarrow CL(X)\) be set-valued mappings. For some \(z \in X\), define the \textit{orbit} \(O(z)\) of \(z\) by

\[
O(z) = \{y \in X : y = z \text{ or } y = f(z) \text{ for some } f \in \Sigma\},
\]
Y being the subsemigroup generated by \( F, G, S \) and \( T \) in the semigroup of all self-mappings on \( X \) with composition operation.

**DEFINITION 1.5.** A point \( z \) is said to be a **common stationary point** of set-valued mappings \( F \) and \( G \) if \( Fz = \{ z \} = Gz \).

2. THE MAIN RESULTS.

Throughout this paper, for any set-valued mapping \( S : X \to CL(X) \), we assume that all the powers of \( S \) map \( X \) into \( CL(X) \). First of all, we prove the following crucial result to be used in the sequel.

**LEMMA 2.1.** Let \( (X,d) \) be a compact metric space and \( S : X \to CL(X) \) be a set-valued mapping such that \( S^i \) is continuous with respect to the generalized Hausdorff distance function \( H \) for some positive integer \( i \). If \( A = \bigcap_{k=1}^\infty S^k(X) \), then \( S(A) = A \).

**PROOF.** Clearly, \( S^{(k+1)}(X) \subseteq S^k(X) \) for \( k = 1, 2, \ldots \). Also, \( x \in X \) implies

\[
S^k x \subseteq A. \tag{1.1}
\]

Let \( y \in A \). Then \( y \in S^{(k+1)}(X) \) for \( k = 1, 2, \ldots \), and so there exists \( x_k \in S^k(X) \) such that \( y = S^k x_k \) for \( k = 1, 2, \ldots \). Since \( X \) is compact, there exists a convergent subsequence \( \{ x_k \} \) of \( \{ x_k \} \) with the limit \( x \). Further, since \( \{ x_j, x_{j+1}, \ldots \} \subseteq S^j(X) \) for \( j = 1, 2, \ldots \), we have \( x \in A \). Also, we have

\[
D(y, S^k x) \leq D(y, S^k x_k) + H(S^k x_k, S^k z).
\]

Letting \( k \to \infty \), we get \( y \in S^k z \). Hence there exist \( x_1, x_2, \ldots, x_k \in X \) such that \( y \in S z \), \( x_1 \in S x_1, \ldots, x_2 \in S x_2, \) and \( x_2 \in S x_2 \). By (1.1), since \( x \in A \), it follows that \( S z \subseteq A \) and so \( x_2 \in A \). A repeated application of (1.1) yields that \( x \in A \). Therefore, we have \( y \in S x \) for some \( z \in A \). Thus, \( A \subseteq S(A) \). From this and (1.1), we conclude that \( S(A) = A \). This completes the proof.

Now, we are in a position to present our main results. We denote

\[
M(x,y,F,G,S,T) = \max\{ h(S^p x, T^q y), \, h(S^p x, F^p z), \, h(T^q y, G^q y), \, h(S^p x, G^q y), \, h(T^q y, F^p z) \}
\]

and

\[
m(x,y,F,G,S,T) = \max\{ h(S^p x, T^q y), \, h(S^p x, G^q y), \, h(T^q y, F^p z) \},
\]

where \( p, q, s \) and \( t \) are positive fixed integers.

**THEOREM 2.1.** Let \( (X,d) \) be a complete metric space and \( F, G, S, T : X \to CL(X) \) be set-valued mappings such that

\[
\begin{align*}
(F, G, S, T) & \text{ and } (FG)^i \text{ are continuous with respect to the distance function } H \text{ for some positive integer } i. \\
\text{Also, } F, G, S \text{ and } T & \text{ are nearly-densifying,} \\
\text{for some } x_0 & \text{, the orbit } O(x_0) \text{ is bounded,} \\
H(F^p x, G^q y) & < M(x,y,F,G,S,T) \\
FG & = GF, (FG)^i S^i = S^i(FG)^i \quad \text{and} \quad (FG)^i T^t = T^t(FG)^i.
\end{align*}
\]

Then \( F, G, S \) and \( T \) have a unique common stationary point \( z \) in \( X \).

**PROOF.** Putting \( A = O(x_0) \), we have clearly \( I(A) = A \) for \( I \in \{ F, G, S, T \} \). Also, the continuity of set-valued mappings \( F, G, S \) and \( T \) yields that \( I(\overline{A}) \subseteq \overline{A} \) for \( I \in \{ F, G, S, T \} \). Further, we have \( A = \{ x_0 \} \cup F(A) \cup G(A) \cup S(A) \cup T(A) \). Thus, \( \alpha(A) = \max\{ \alpha(x_0), \alpha(F(A)), \alpha(G(A)), \alpha(S(A)), \alpha(T(A)) \} \) and also \( \overline{A} \) is compact. Now, define \( B = \bigcap_{n=1}^\infty (FG)^n(\overline{A}) \). Then \( B \) is compact. By Lemma 2.1, \( (FG)(B) = B \) and the condition (2.4) ensures that \( F(B) = B = G(B), \, S^i(B) \subseteq B \) and \( T^t(B) \subseteq B \). Since \( B \) is compact, there exist \( x_1, x_2 \in B \) such that \( d(x_1, x_2) = \sup\{ d(x, y) : x, y \in B \} = \delta(B) \), say. Also, there exist \( w_1, w_2 \in B \) such that \( x_1 \in F^p w_1 \) and \( x_2 \in G^q w_2 \). Suppose that \( \delta(B) > 0 \). Then, by (2.3), we
have

$$\delta(B) = d(x_1, x_2) \leq H(F^{i}w_1, G^{i}w_2)$$

$$< M(w_1, w_2, F^{i}, G^{i}, S^{i}, T^{i})$$

$$\leq \delta(B),$$

which is a contradiction. Thus, $\delta(B) = 0$ and hence $B = \{z\}$, say. Therefore, $z$ is a common stationary point of $F, G, S$ and $T$. The uniqueness of $z$ follows from condition (2.3). This completes the proof.

**THEOREM 2.2.** Let $(X,d)$ be a compact metric space and $F, G, S, T : X \to CL(X)$ be set-valued mappings such that

(2.5) $(FG)^i$ is continuous for some positive integer $i$,

(2.6) $H(F^{i}z, G^{i}y) < M(z, y, F^{i}, G^{i}, S^{i}, T^{i})$ whenever the left-hand side is positive,

(2.7) $FG = GF, (FG)^i = S^{i}(FG)^i$ and $(FG)^iT^{i} = T^{i}(FG)^i$.

Then $F, G, S$ and $T$ have a unique common stationary point $z$ in $X$. Further, $z$ is the unique common stationary point of $F$ and $G$.

**PROOF.** If we put $B_n = (FG)^n(X)$, as in the proof of Theorem 2.1, we have $B = \{z\}$ and $z$ is a unique common stationary point of $F, G, S$ and $T$. Since any common stationary point of $F$ and $G$ is a point of $B = \{z\}$, it follows that $z$ is the unique common stationary point of $F$ and $G$. This completes the proof.

**REMARK.** Theorem 2 of Fisher [2] and theorems in Fisher [3] follow as corollaries of our Theorem 2.2. In fact, our theorem can be regarded as an improvement over the above theorems due to Fisher.

**THEOREM 2.3.** Let $(X,d)$ be a complete metric space and $F, G, S, T : X \to CL(X)$ be set-valued mappings such that

(2.8) $F, G, S, T, F^i$ and $G^j$ are continuous with respect to the distance function $H$ for some positive integers $i$ and $j$. Also, $F, G, S$ and $T$ are nearly-densifying,

(2.9) for some $z_0 \in X$, the orbit $O(z_0)$ is bounded,

(2.10) $H(F^{i}z_0, G^{i}y) < M(z_0, y, F^{i}, G^{i}, S^{i}, T^{i})$ whenever the left-hand side is positive,

(2.11) $S^{i}F^i = F^iS^{i}$ and $T^{i}G^i = G^iT^i$.

Then $F, G, S$ and $T$ have a unique common stationary point $z$ in $X$.

**PROOF.** Let $A = O(z_0)$. Then as in the proof of Theorem 2.1, $A$ is compact. If we define

$$B = \cap_{n=1}^{\infty} F^n(A) \text{ and } K = \cap_{n=1}^{\infty} G^n(A),$$

by Lemma 2.1, $F(B) = B$ and $G(K) = K$. Also, it follows that $B$ and $K$ are compact subsets of $X$. By the condition (2.11), also we have $S^n(B) \subseteq B$ and $T^n(K) \subseteq K$. Then, there exist $x_1, w_1 \in B$ and $y_1, w_2 \in K$ such that

$$d(x_1, y_1) = \sup \{d(x, y) : x \in B, y \in K\} = \delta(B, K),$$

with $x_1 \in F^{i}w_1$ and $y \in G^{i}w_2$. Suppose that $\delta(B, K) > 0$. Then, by the condition (2.10), we have

$$\delta(B, K) = d(x_1, y_1)$$

$$\leq H(F^{i}w_1, G^{i}w_2)$$

$$< M(w_1, w_2, F^{i}, G^{i}, S^{i}, T^{i})$$

$$\leq \delta(B, K),$$
which is a contradiction. Therefore, \( \kappa(B,K) = 0 \) and \( B = K = \{ z \} \). Thus \( z \) is a common stationary point of \( F,G,S \) and \( T \). The uniqueness of \( z \) follows easily from the condition (2.10). This completes the proof.

**THEOREM 2.4.** Let \((X,d)\) be a compact metric space and \( F,G,S,T : X \rightarrow CL(X) \) be set-valued mappings such that

\[
F' \text{ and } G' \text{ are continuous with respect to the distance function } H \text{ for some positive integers } i \text{ and } j,
\]

\[
H(F^i x, G^j y) < m(x,y,F^i,G^j,S^i,T^j) \quad \text{whenever the left-hand side is positive,}
\]

\[
F^iS^i = S^iF^i \text{ and } G^jT^j = T^jG^j.
\]

Then \( F,G,S \) and \( T \) have a unique common stationary point \( z \) in \( X \). Further, \( z \) is the unique common stationary point of the pairs \( F,S \) and \( G,T \). Also, \( z \) is the unique common stationary point of \( F \) and \( G \).

**PROOF.** Let \( B = \cap_{n=1}^{\infty} F^n(X) \) and \( K = \cap_{n=1}^{\infty} G^n(X) \). Then as in the proof of Theorem 2.3, we get \( B = K = \{ z \} \) and \( z \) is a unique common stationary point of \( F,G,S \) and \( T \). Since any stationary point of \( F \) is a point of \( B = \{ z \} \) and any stationary point of \( G \) is a point of \( K = \{ z \} \), it follows that \( z \) is the unique stationary point of \( F \) as well as of \( G \). This completes the proof.

**REMARK.** Theorem 5 of Fisher [1] follows as a corollary of our Theorem 2.3.

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REFERENCES

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