ON CERTAIN CLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. A function f , analytic in the unit disk E and given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is said to be in the family K_n if and only if $D^n f$ is close-to-convex, where $D^n f = \frac{z}{(1-z)^{n+1}} * f$, $n \in N_0 = \{0, 1, 2, \dots\}$ and $*$ denotes the Hadamard product or convolution. The classes K_n are investigated and some properties are given. It is shown that $K_{n+1} \subseteq K_n$ and K_n consists entirely of univalent functions. Some closure properties of integral operators defined on K_n are given.

KEY WORDS AND PHRASES. Univalent, close-to-convex, starlike, convolution, integral operators.

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1. INTRODUCTION.

Let A denote the class of functions $f: f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ analytic in the unit disk $E = \{z: |z| < 1\}$. The Hadamard product or convolution of two functions $f, g \in A$ is denoted by $f * g$. For $n \in N_0 = \{0, 1, 2, 3, \dots\}$, let $D^n f = (\frac{z}{(1-z)^{n+1}} * f)$, so that

$$D^n f = z(z^{n-1} f)^{(n)} / n!.$$

Let $S \subset A$ be the class of univalent functions and for $0 \leq \beta < 1$, let $C(\beta)$ and $S^*(\beta)$ denote the subclasses of S consisting of convex functions of order β and starlike functions of order β respectively. The classes C and S^* of convex and starlike functions, respectively, are identified by $C(0) \equiv C$ and $S^*(0) = S^*$.

A function $f \in S$ belongs to the class $K(\alpha, \beta)$ of close-to-convex of order α and type β if and only if for some $g \in S^*(\beta)$ and $0 \leq \alpha < 1$,

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \alpha, \quad z \in E.$$

It is clear that $K(0, 0) \equiv K$, the class of close-to-convex univalent functions [1].

DEFINITION 1.1. For $n \in N_0$, a function $f \in A$ is said to belong to the classes R_n , if and only if for $z \in E$,

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > 0. \quad (1.1)$$

Thus $R_0 \equiv S^*$ and $R_1 \equiv C$. In [2], Ahuja discussed these classes and showed that $R_{n+1} \subset R_n$ for

each $n \in N_0$. This implies that functions in R_n are starlike and hence univalent.

We now extend the classes R_n , as follows:

DEFINITION 1.2. Let $f \in A$. Then $f \in K_n$ if and only if there exists $g \in R_n$ such that for $z \in E$,

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n g(z)} > 0. \quad (1.2)$$

We note that $K_0 \equiv K$ and $K_1 \equiv C^*$, the class of quasi-convex functions introduced in [3].

In order to develop some results for K_n , we shall need the following:

LEMMA 1.1 [4]. Let w be analytic in E . If $|\omega|$ assumes its maximum value on the circle $|z| = r$ at a point z_0 , then

$$z_0 \omega'(z_0) = k \omega(z_0),$$

where $k \geq 1$.

LEMMA 1.2 [5]. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\psi(u, v)$ be a complex-valued function satisfying the conditions:

- (i) $\psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,
- (ii) $(1, 0) \in D$ and $\psi(1, 0) > 0$,
- (iii) $\operatorname{Re}(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$.

If $h(z) = 1 + \sum_{k=2}^{\infty} c_k z^k$ is a function analytic in E , such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re} \psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$ in E .

LEMMA 1.3. [6]. Let ϕ be convex and g be starlike in E . Then, for F analytic in E with $F(0) = 1$, $\frac{\phi^* F g}{\phi^* g}$ is contained in the convex hull of $F(E)$.

2. PROPERTIES OF THE FAMILY K_n .

We first prove that all functions in K_n are close-to-convex and hence univalent.

THEOREM 2.1. $K_{n+1} \subset K_n$, for each $n \in N_0$.

PROOF. Let $f \in K_{n+1}$. Then for $z \in E$

$$\operatorname{Re} \frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} > 0, \text{ for some } g \in R_{n+1}.$$

Define $\omega(z)$ in E such that

$$\frac{z(D^n f(z))'}{D^n g(z)} = \frac{1 - \omega(z)}{1 + \omega(z)}, \quad (2.1)$$

where $\omega(0) = 0$ and $\omega(z) \neq -1$. We show that $|\omega(z)| < 1$.

From (2.1) we have

$$z(D^n f(z))' = D^n g(z) \cdot \frac{1 - \omega(z)}{1 + \omega(z)}. \quad (2.2)$$

So, from (2.2) and the identity

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z), \quad (2.3)$$

we obtain

$$z(D^{n+1} f(z))' = \frac{1}{n+1} \left[z(D^n g(z))' \frac{1 - \omega(z)}{1 + \omega(z)} + D^n g(z) \left\{ \frac{-2z\omega'(z)}{(1 + \omega(z))^2} + n \frac{1 - \omega(z)}{1 + \omega(z)} \right\} \right]. \quad (2.4)$$

Now apply (2.3) for the function g , and use (2.4) to obtain

$$\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} = \frac{1 - \omega(z)}{1 + \omega(z)} + \frac{1}{n+1} \frac{D^n g(z)}{D^{n+1} g(z)} \cdot \left[\frac{-2z\omega'(z)}{(1 + \omega(z))^2} \right]. \quad (2.5)$$

Since $R_{n+1} \subset R_n$, this implies that $g \in R_n$ and hence there exists an analytic function $\omega_1(z)$ with $\omega_1(0) = 0$ and $|\omega_1(z)| < 1$ such that

$$\frac{D^{n+1}g(z)}{D^n g(z)} = \frac{1 - \omega(z)}{1 + \omega(z)}. \quad (2.6)$$

Thus using (2.6) in (2.5) we have

$$\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} = \frac{1 - \omega(z)}{1 + \omega(z)} + \frac{1}{n+1} \left(\frac{1 + \omega_1(z)}{1 - \omega_1(z)} \right) \left(\frac{-2z\omega'(z)}{(1 + \omega(z))^2} \right). \quad (2.7)$$

Suppose now that for $z \in E$

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1, (\omega(z_0) \neq -1).$$

Then it follows, from Lemma 1.1, that

$$z_0 \omega'(z_0) = k \omega(z_0),$$

where $k \geq 1$.

Setting $\omega(z_0) = e^{i\Theta}$ and $\omega_1(z_0) = re^{i\phi}$ in (2.7) gives

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)} \right\} &= \operatorname{Re} \left[\left(\frac{1}{n+1} \right) \frac{-2k(e^{i\Theta} + e^{-i\Theta} + 2)(1 + r^2 + 2r \cos \phi)}{|1 + re^{i\phi}|^2 |1 + e^{i\Theta}|^2} \right] \\ &= \frac{-4k}{n+1} \left[\frac{(\cos \Theta + 1)(1 + r^2 + 2r \cos \phi)}{|1 + re^{i\phi}|^2 |1 + e^{i\Theta}|^2} \right] \end{aligned}$$

Hence, if $\phi = \frac{\pi}{2}$,

$$\operatorname{Re} \frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)} < 0,$$

where $g \in R_{n+1}$ and $k \geq 1$. This contradicts our hypothesis that $f \in K_{n+1}$. Thus $|\omega(z)| < 1$ and so $f \in K_n$.

From Theorem 2.1, we note that $f \in K_n$ implies that $f \in K$ and so f is univalent in E . Also, since $K_n \subset K_1 \equiv C^*$ it follows that f is quasi-convex.

REMARK 2.1. Let $f \in K_n$ and be given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then

$$\begin{aligned} D^n f(z) &= \frac{z}{(1-z)^{n+1}} * f(z), \\ &= \left[z + \sum_{k=2}^{\infty} \frac{(k+n-1)!}{n!(k-1)!} z^k \right] * \left[z + \sum_{k=2}^{\infty} a_k z^k \right], \\ &= z + \sum_{k=2}^{\infty} \frac{(k+n-1)!}{n!(k-1)!} a_k z^k. \end{aligned} \quad (2.8)$$

Thus from (2.8) and Definition 1.2 it follows that

$$f \in K_n \text{ if and only if } D^n f \in K. \quad (2.9)$$

THEOREM 2.2. Let $f \in K_n$ and be given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then for $k \geq 2$ and $n \geq 0$,

$$|a_k| \leq \frac{(k!)(n!)}{(k+n-1)!}.$$

This result is sharp with equality for the function f_0 , where

$$D^n f_0(z) = \frac{z}{(1-z)^2}. \quad (2.10)$$

The proof follows immediately from (2.8), (2.9), and the well-known coefficient result for the class K of close-to-convex functions.

THEOREM 2.3. (Covering theorem). Let $f \in K_n$. If B is the boundary of the image of E under f , then every point of B is distance at least $\frac{n+1}{2(n+2)}$ from the origin.

PROOF. Let $f(z) \neq c, c \neq 0$. Then f_1 given by

$$f_1(z) = \frac{cf(z)}{c-f(z)}$$

is univalent in E . Write $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\frac{cf(z)}{c-f(z)} = z + (a_2 + \frac{1}{c})z^2 + \dots$$

and, since $f_1 \in S$, it follows that

$$|a_2 + \frac{1}{c}| \leq 2.$$

Hence,

$$|\frac{1}{c}| \leq 2 + |a_2|,$$

and using Theorem 2.2, we obtain

$$|c| \geq \frac{n+1}{2(n+2)}.$$

This completes the proof.

We note that when $n=0$, $|c| \geq \frac{1}{4}$ and when $n=1$, $|c| \geq \frac{1}{2}$ (see [3] and [7]).

THEOREM 2.4. $\bigcap_{n=0}^{\infty} K_n = \{id\}$,

where id is the identity function z .

PROOF. Let $f(z) = g(z) = z$ in (1.2), then it follows trivially that $z \in K_n$ for $n \geq 0$.

On the contrary, assume that $f \in \bigcap_{n=0}^{\infty} K_n$ with $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

Then it follows from Theorem 2.2. that $f(z) = z$.

3. INTEGRAL OPERATORS.

Let the operator $I_\lambda: A \rightarrow A$ be defined by $f = I_\lambda(F)$, as

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^{\frac{z}{\lambda}} \xi^{\frac{1}{\lambda}-2} F(\xi) d\xi, \quad (3.1)$$

where $0 < \lambda \leq 1$.

For $\lambda = \frac{1}{2}$, Libera [8] established that the operator

$$I(F) = \frac{2}{z} \int_0^{\frac{z}{2}} F(\xi) d\xi$$

preserves convexity, starlikeness, and close-to-convexity. Bernardi [9] greatly generalized Libera's results. Many authors have studied the operators of the form (3.1), see e.g. [7]. Ahuja [2] has discussed the $\lambda = \frac{1}{\gamma+1}$, γ complex and $\operatorname{Re} \gamma \neq -1$ for the classes R_n . Here we shall consider (3.1) for K_n .

We shall need the following [2]:

Let $I_\lambda: A \rightarrow A$ be defined by (3.1) with $0 < \lambda \leq 1$. If $F \in R_n$, then $I_\lambda(F) \in R_n(\alpha)$, i.e., for $z \in E$

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \alpha,$$

where $0 < \alpha < 1$ and

$$\alpha = \frac{1}{4\lambda} \left[-(2-\lambda) + \sqrt{9\lambda^2 + 4\lambda + 1} \right] \quad (3.2)$$

We now prove:

THEOREM 3.1. Let $F \in K_n$ and let $f = I_\lambda(F)$ be defined as in (3.1) for $0 < \lambda \leq 1$. Then $f \in K_n(\beta, \alpha)$, where α is given by (3.2) and $\beta(0 \leq \beta < 1)$ is defined by (3.9).

PROOF. Let $G \in R_n$ and $I_\lambda(G) = g$, where I_λ is defined by (3.1). So that

$$D^n G(z) = (1-\lambda)D^n g(z) + \lambda z(D^n g(z))' \quad (3.3)$$

and

$$D^n F(z) = (1-\lambda)D^n f(z) + \lambda z(D^n f(z))',$$

Set

$$\frac{z(D^n f(z))'}{D^n g(z)} = (1-\beta)p(z) + \beta, \quad (3.4)$$

where

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

We need to show that $\operatorname{Re} p(z) > 0$ for $z \in E$.

From (3.3), we have

$$\frac{z(D^n F(z))'}{D^n G(z)} = \frac{(1-\lambda) \frac{z(D^n f(z))'}{D^n g(z)} + \lambda z \frac{z(D^n f(z))'}{D^n g(z)}}{(1-\lambda) + \lambda z \frac{(D^n g(z))'}{D^n g(z)}} \quad (3.5)$$

Since $g \in R_n(\alpha)$, where α is given by (3.2), we can write

$$\frac{z(D^n g(z))'}{D^n g(z)} = (1-\alpha)p_0(z) + \alpha, \quad (3.6)$$

where $\operatorname{Re} p_0(z) > 0, z \in E$.

Also, from (3.4) and (3.6), we obtain

$$\begin{aligned} \frac{z(D^n f(z))'}{D^n g(z)} &= \left\{ \frac{z(D^n g(z))'}{D^n g(z)} ((1-\beta)p(z)) + \beta \frac{z(D^n g(z))'}{D^n g(z)} + (1-\beta)zp'(z) \right\}, \\ &= [(1-\alpha)p_0(z) + \alpha][(1-\beta)p(z) + \beta] + (1-\beta)zp'(z). \end{aligned} \quad (3.7)$$

Using (3.4), (3.6), and (3.7) in (3.5), it follows that

$$\frac{z(D^n F(z))'}{D^n G(z)} = \beta + (1-\beta)p(z) + \frac{\lambda(1-\beta)zp'(z)}{(1-\lambda) + \lambda[(1-\alpha)p_0(z) + \alpha]}. \quad (3.8)$$

Next define $\psi(u, v)$ by taking $u = p(z)$ and $v = zp'(z)$ in (3.8) by

$$\psi(u, v) = \beta + (1-\beta)u + \frac{\lambda(1-\beta)v}{\lambda(1-\alpha)p_0 - \lambda(1-\alpha) + 1}.$$

It is clear that $\psi(u, v)$ satisfies conditions (i) and (ii) of Lemma 1.2. To verify condition (iii), we note that

$$\operatorname{Re} \psi(iu_2, v_1) = \beta + \frac{\lambda(1-\beta)v_1\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}}{\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2},$$

where $p_0(z) = p_1 + ip_2$, p_1, p_2 being functions of x and y and $\operatorname{Re} p_0 = p_1 > 0$.

By putting $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we obtain

$$\operatorname{Re} \psi(iu_2, v_1) \leq \beta - \frac{\lambda(1-\beta)(1+u_2^2)[\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1]}{2\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2} = \frac{A + Bu_2^2}{2C},$$

where

$$C = \{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2 > 0,$$

$$A = 2\beta\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2 - \lambda(1-\beta)[\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1],$$

and

$$B = -\lambda(1-\beta)[\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1].$$

We note that $\operatorname{Re} \psi(iu_2, v_1) \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we obtain $\beta \leq \beta_A$ where

$$\beta_A = \frac{\lambda^2(1-\alpha)^2 p_2^2 + \lambda[\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1]}{2\{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}^2 + \lambda^2(1-\alpha)^2 p_2^2 + \{\lambda(1-\alpha)p_1 - \lambda(1-\alpha) + 1\}} \geq 0. \quad (3.9)$$

Also, from $B \leq 0$, we have $\beta_A < 1$ and the condition (iii) is satisfied to give $\operatorname{Re} p(z) > 0$ for $z \in E$ which implies that $f \in K_n(\beta, \alpha)$.

If we put $n = \left(\frac{1}{\beta} - 1\right)$ in (3.1) we have the following:

THEOREM 3.2. Let $F \in K_n$ and let

$$f(z) = (n+1)z^{-n} \int_0^z \xi^{n-1} F(\xi) d\xi. \quad (3.10)$$

Then $f \in K_{n+1}$.

$$\text{PROOF. Let } g(z) = (n+1)z^{-n} \int_0^z \xi^{n-1} G(\xi) d\xi, \quad (3.11)$$

where $G \in R_n$. Then from [2] $g \in R_{n+1}$. From (3.10) and (3.11) we have

$$D^n F(z) = \frac{n}{n+1} D^n f(z) + \frac{1}{n+1} z(D^n f(z))' \quad (3.12)$$

and

$$D^n G(z) = \frac{n}{n+1} D^n g(z) + \frac{1}{n+1} z(D^n g(z))'. \quad (3.13)$$

From (3.12) and the identity

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - nD^n f(z),$$

we have

$$D^n F(z) = D^{n+1}f(z). \quad (3.14)$$

Since $F \in K_n$, it follows that, for $z \in E$,

$$\operatorname{Re} \left(\frac{z(D^n F(z))'}{D^n G(z)} \right) > 0, \quad G \in R_n$$

and thus, using (3.14), we conclude that, for $z \in E$

$$\operatorname{Re} \left[\frac{z(D^n + 1)f(z)}{D^n + 1g(z)} \right] > 0 \text{ for } g \in R_{n+1}.$$

THEOREM 3.3. Let $f \in K_n, n \geq 0$ and $\phi \in C$. Then $\phi^*f \in K_n$.

PROOF. First we prove that, if $g \in R_n$, then $(\phi^*g) \in R_n$. It is sufficient to show that $D^n(\phi^*g) \in S^*$.

Now

$$D^n(\phi^*g)(z) = \frac{z}{(1-z)^{n+1}} * (g^*\phi)(z) = \phi(z) * \frac{z}{(1-z)^{n+1}} * g(z) = (\phi^*D^n g)(z).$$

Since $g \in R_n$ and $\phi \in C$, it follows that $\phi * g \in R_n$, see [2].

Next, we prove that $(\phi^*f) \in K_n$.

$$\frac{z[D^n(\phi^*f)(z)]'}{D^n(\phi^*f)(z)} = \frac{z[\phi(z)^*D^n f(z)]'}{\phi(z)^*D^n g(z)} = \frac{\phi(z)^* \frac{(D^n f(z))'}{D^n g(z)} \cdot D^n g(z)}{\phi(z)^*D^n g(z)}$$

Applying Lemma 1.3 with $F(z) = \frac{z(D^n f(z))'}{D^n g(z)}$, $D^n g(z) \in S^*$ and $\operatorname{Re} R(z) > 0$, we obtain

$$\operatorname{Re} \frac{z[D^n(\phi^*f)(z)]'}{D^n(\phi^*f)(z)} > 0 \text{ for } z \in E.$$

This proves Theorem 3.3.

REMARK 3.1. Theorem 3.3 is an analogue of the Polya-Schoenberg conjecture [6] for the family K_n . Many results on K_n can be deduced as applications.

We give the following:

THEOREM 3.4. Let $f \in K_n, n \geq 0$ and be defined by (3.1). Then $F \in K_n, n \geq 0$ for $|z| < r_0$, where r_0 is given by

$$r_0 = \frac{1}{(2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1})}. \quad (3.15)$$

The function f_0 , defined by (2.10), shows that this result is sharp.

PROOF. Let $\phi_\lambda(z) = \sum_{k=1}^{\infty} [\lambda(k-1)+1]z^k, 0 < \lambda < 1$.

Then $\lambda_\phi \in C$ for $|z| < r_0$ where r_0 is given by (3.15). Also $F(z) = (\phi_\lambda^*f(z))$ and so using Theorem 3.3, we see that $F \in K_n, n \geq 0$ for $|z| < r_0$.

REMARK 3.2. We note that Theorem 3.3 shows that the family K_n is invariant under the following integral operators

$$I_1(f) = \int_0^z \frac{f(\xi)}{\xi} d\xi = (f^*\phi_1)(z),$$

$$I_2(f) = \frac{2}{\pi} \int_0^z \frac{f(\xi)}{\xi} d\xi = (f^*\phi_2)(z), \quad (\text{Libera's operator})$$

$$\begin{aligned} I_3(f) &= \int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi} d\xi, \quad |x| < 1, x \neq 1, \\ &= (f^*\phi_3)(z) \end{aligned}$$

and

$$I_4(f) = \frac{1+c}{c} \int_0^z \xi^{c-1} f(\xi) d\xi, \quad \operatorname{Re}(c) > 0$$

$$= (f^* \phi_4)(z),$$

where $\phi_i \in C, i = 1, 2, 3, 4$
and

$$\phi_1(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z),$$

$$\phi_2(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n = \frac{-2[z + \log(1-z)]}{z},$$

$$\phi_3(z) = \sum_{n=1}^{\infty} \frac{1-x^n}{n(1-x)} z^n = \frac{1}{1-x} \log \frac{1-xz}{1-z}, \quad |x| = 1, x \neq 1,$$

and

$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \quad \operatorname{Re}(c) > 0.$$

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