NOTE ON POINTWISE CONTRACTIVE PROJECTIONS

L.M. SANCHEZ RUIZ and J.R. FERRER VILLANUEVA
Departamento de Matemática Aplicada
Universidad Politécnica de Valencia
46071 Valencia
Spain

(Received April 16, 1992)

ABSTRACT: Let \( C(X) \) be the space of real-valued continuous functions on a Hausdorff completely regular topological space \( X \), endowed with the compact-open topology. In this paper necessary and sufficient conditions are given for a subspace of \( C(X) \) to be the range of a pointwise contractive projection in \( C(X) \).

KEY WORDS AND PHRASES: Contractive projection, extreme point.

Several authors have considered the problem of characterizing the subspaces of \( C(K) \) admitting contractive projections, \( K \) being a compact Hausdorff space (cf. Lindestrauss [1], Lindestrauss-Wulbert [2] and Lindberg [3]). And when \( X \) is a Hausdorff completely regular topological space with a fundamental sequence of compact sets, we have discussed in [4] conditions for a subspace \( E \) of \( C(X) \) to be the range of a compact contractive projection in \( C(X) \). In this note we want to study the problem that arises when the projection \( p \) of \( C(X) \) onto \( E \) is pointwise contractive, i.e., when for each \( x \in X \) it is \( |p(f)(x)| \leq |f(x)| \) for every \( f \in C(X) \).

Hereafter \( X \) will stand for any Hausdorff completely regular topological space and \( C(X) \) for the space of the continuous real-valued functions on \( X \) endowed with the compact-open topology. Given a linear subspace \( E \) of \( C(X) \) and \( x \in X \), we set \( E_x = \{ f \in E : |f(x)| \leq 1 \} \) and \( C_x = \{ f \in C(X) : |f(x)| \leq 1 \} \), holding \( E_x^* \) and \( C_x^* \) for their polar sets in the topological dual spaces of \( E \) and \( C(X) \), respectively. \( E \) is called separating if for each \( x, y \in X \), \( x \neq y \), there is some \( f \in E \) such that \( f(x) \neq f(y) \). For each \( x \in X \), \( \delta_x \) will denote the linear form of \( (C(X))^* \) such that \( \delta_x(f) = f(x) \forall f \in C(X) \). If \( A \) is a subset of \( (C(X))^* \), \( x \) is called an extreme point of \( A \) if \( x \neq \lambda x + (1-\lambda)y \) with \( 0 < \lambda < 1 \), \( x, y \in A \), implies that \( z = x + y \). Given \( x \in X \), \( x \) is a double point of \( E \) if there is some \( y_x \in X \) such that \( f(x) + f(y_x) = 0 \) for every \( f \in E \). We shall say \( x \) is an autodouble point if \( f(x) = 0 \) for every \( f \in E \), i.e., if \( x \) is a double point and \( y_x = x \). If \( x \) is not a double point, \( x \) is called a single point. If \( E \) is separating and \( x \) is a double point then \( y_x \) is unique, there being at most only one autodouble point. Clearly, there are no double points if \( E \) contains the constant functions.

LEMMA. Let \( E \) be a separating linear subspace of \( C(X) \). For each \( x \in X \), \( \pm \delta_x \) are the only extreme points of \( E_x^* \).

PROOF. Clearly the \( \sigma(E,E) \)-closed convex cover of \( F = (\delta_x, -\delta_x) \) is contained in \( E_x^* \) and if \( \varphi \notin C_F^* \) there must be some \( f \in E \) such that \( \varphi(f) > 1 \) and \( |f(x)| \leq 1 \), i.e., \( \varphi \notin E_x^* \). On the other hand \( E_x^* \) has some extreme point, since it is weakly compact, and it will be contained in \( F \). Hence \( \pm \delta_x \) are the only extreme points of \( E_x^* \).
PROPOSITION. Let $E$ be a separating subspace of $C(X)$ and $p$ a pointwise contractive projection of $C(X)$ onto $E$. Then for each $x \in X$, the transpose linear mapping $p^*$ of $p$ satisfies:

i) $p^*(\delta_x) = 0$ if $x$ is an autodouble point.

ii) $p^*(\delta_x) = \delta_x$ if $x$ is not an autodouble point.

PROOF. i) For each $f \in C(X)$, $p^*\delta_x(f) = \delta_x(pf) = (pf)(x) = 0$ since $pf \in E$. Hence $p^*(\delta_x) = 0$.

ii) Let $E(x)$ be the set of all $\varphi \in C^*_{x}$ such that $\varphi = \delta_x(f)$ for some $f \in C_x$ where $C_x$ is the closed convex hull of the extreme points of $C^*_{x}$. By Krein-Millman's theorem, $E(x)$ coincides with the closed convex hull of its extreme points. Now $p^*\delta_x \in E(x)$ since for each $g \in C^*_x$, $\|p^*\delta_x(g)\| = \|\delta_x(pg)\| = \|pg(x)\|$ is an autodouble point.

Let $E(x) = \{\varphi \in C^*_x \mid \varphi = \delta_x(f)\}$ and $\delta_x = p^*\varphi$. By the closed graph theorem, $E(x)$ is a closed linear subspace of $C^*_x$. Moreover, for each $x \in X$, $\delta_x = \delta_x(f)$ for some $f \in C_x$.

THEOREM 1. Let $E$ be a separating subspace of $C(X)$ and $p$ a pointwise contractive projection of $C(X)$ onto $E$. Then:

i) If for each $x \in X$ there is some $f \in E$ such that $f(x) \neq 0$, every point of $X$ is single.

ii) If there is some $x \in X$ such that $f(x) = 0$ for each $f \in E$, $x$ is the only double point of $X$. Moreover, $x$ is isolated and, clearly, autodouble.

THEOREM 2. A locally convex topological vector space $E$ is isomorphic to the range of a pointwise contractive projection in $C(X)$ if and only if $E$ is isomorphic to either $C(X)$ or some $C^*_x$.

PROOF. Assume $E$ is separating. If each point $x \in X$ is single, $p^*\delta_x = 0$ for every $x \in X$. On the other hand, if there exists some double point $x$, $E$ is contained in $C^*_x$. But for each $x \in X$, $p^*\delta_x = 0$ and $p^*\delta_x(f) = f(x)$ for $x \neq x$. If $E$ is not separating, we are able to form the quotient by identifying those points which are not separated by $E$ and the same conclusion yields.

Conversely, if $E$ is isomorphic to some $C^*_x$, then the mapping $p : C(X) \rightarrow C^*_x$ defined by $p(f) = f_x$, where $f_x(x) = f(x)$ for $x \neq x$ and $f_x(x) = 0$, is pointwise contractive and $p^* = f$ for each $f \in C^*_x$. (Refer to [1], [2], [3], [4], [5])

REFERENCES


Submit your manuscripts at http://www.hindawi.com