ON TOTALLY UMBILICAL CR-SUBMANIFOLDS OF A KAHLER MANIFOLD

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(Received June 4, 1991 and in revised form April 21, 1992)

ABSTRACT. Let $M$ be a compact 3-dimensional totally umbilical CR-submanifold of a Kaehler manifold of positive holomorphic sectional curvature. We prove that if the length of the mean curvature vector of $M$ does not vanish, then $M$ is either diffeomorphic to $S^3$ or $RP^3$ or a lens space $L^3_{p,q}$.


1991 AMS SUBJECT CLASSIFICATION CODES. Primary 53C40. Secondary 53C55.

1. INTRODUCTION.

Totally umbilical CR-submanifolds of a Kaehler manifold have been considered by Bejancu [2], Blair, and Chen [3]. Recently Deshmukh and Husain [5] have also studied these submanifolds. In fact, they have proved a classification theorem when the dimension of the submanifold $M$ is $\geq 5$. In this paper we consider 3-dimensional totally umbilical CR-submanifolds of a Kaehler manifold. For this case we have obtained the following theorem:

THEOREM 1.1. Let $M$ be a compact 3-dimensional totally umbilical CR-submanifold of a Kaehler manifold $M$, of positive holomorphic sectional curvature. If the length of the mean curvature vector of $M$ does not vanish then $M$ is diffeomorphic either to $S^3, RP^3$ or the lens space $L^3_{p,q}$.

2. PRELIMINARIES.

Let $\tilde{M}$ be an $m$-dimensional Kaehler manifold with almost complex structure $J$. A $(2p+q)$-dimensional submanifold $M$ of $\tilde{M}$ is called a CR-submanifold if there exists a pair of orthogonal complementary distributions $D$ and $\tilde{D}$ such that $JD = D$ and $J\tilde{D} \subset \nu$, where $\nu$ is the normal bundle of $M$ and $\dim \tilde{D} = q[1]$. Thus the normal bundle $\nu$ splits as $\nu = J\tilde{D} \oplus \mu$, where $\mu$ is invariant sub-bundle of $\nu$ under $J$. A CR-submanifold is said to be proper if neither $D = \{0\}$ nor $\tilde{D} = \{0\}$.

We denote by $\nabla, \bar{\nabla}, \check{\nabla}$ the Riemannian connection on $\tilde{M}, M$ and the normal bundle respectively. They are related by

\[ \bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \] \hspace{1cm} (2.1)
\[ \check{\nabla}_X N = -A_N X + \bar{\nabla}_X N, \hspace{1cm} N \in \nu \] \hspace{1cm} (2.2)
where \( h(X,Y) \) and \( A_N X \) are the second fundamental forms which are related by

\[
g(h(X,Y), N) = g(A_N X, Y) \tag{2.3}
\]

Now a \( CR \)-submanifold is said to be totally umbilical if

\[
h(X,Y) = g(X,Y)H
\]

where \( H = \frac{1}{k} \) \( (\text{trace } h) \) is the main curvature vector. If \( M \) is totally umbilical \( CR \)-submanifold, then equations (2.1) and (2.2) become

\[
\nabla_X Y = \nabla_X Y + g(X,Y)H
\tag{2.4}
\]

\[
\nabla_X N = - g(H, N)X + \frac{1}{h} \nabla_X N \tag{2.5}
\]

For \( X, Y, Z, W \in X(M) \), the equation of Gauss is given by

\[
R(X, Y; Z, W) = R(X, Y; Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \tag{2.6}
\]

3. 3-DIMENSIONAL CR-SUBMANIFOLD OF A KAEHLER MANIFOLD.

(A) Let \( M \) be a compact totally umbilical 3-dimensional \( CR \)-submanifold of a Kaeehler manifold \( \tilde{M} \). If \( \dim D = 0 \), then \( M \) will be totally real. Therefore, we assume that \( \dim D \neq 0 \). Since \( M' \) is 3-dimensional it follows that \( \dim D = 2 \). We can then choose a frame field \( \{X, JX, Z\} \) on \( M \), where \( X \notin D \) and \( Z \notin D \). We first have the following:

**Lemma 1.** Let \( \{X, JX, Z\} \) be a frame field on \( M \), \( X \notin D, Z \notin D \). Then \( \nabla Z Z = 0 \) and \( H \notin J D \).

**Proof.** Using (2.4) and (2.5) in the equation \( \nabla Z Z = J \nabla Z Z \), we obtain

\[
-g(H, JZ)JZ + \frac{1}{h} \nabla Z Z = - \nabla Z Z - h(Z,Z) \tag{3.1}
\]

Taking inner product in (3.1) with \( W \in D \) we have

\[
g(\nabla Z Z, W) = 0 \tag{3.2}
\]

From (3.2) we have \( \nabla Z Z \notin J D \). Since \( g(Z, Z) = 1 \), we also have \( \nabla Z Z \notin D \). Therefore \( \nabla Z Z = 0 \).

Now for \( X, Y \neq 0 \) in \( D \) we use (2.4) and the equation \( J \nabla X Y = \nabla X J Y \) to get

\[
J \nabla X Y + g(X,Y)JH = \nabla X J Y + g(X,Y)H \tag{3.3}
\]

Taking inner product in (3.1) with \( N \in \mu \) we have

\[
g(X,Y)g(JH, N) = g(X,Y)g(H, N) \tag{3.4}
\]

In particular if we let \( Y = JX \) in (3.4) we get

\[
\|X \parallel g(H, N) = 0, \quad N \in \mu \quad \text{Therefore } H \in J \tilde{D}. \tag{3.5}
\]

Consider the frame field \( \{X, JX, Z\} \) on \( M \). Since \( M \) is totally umbilical the equation \( h(Y, W) = g(Y, W)H \) for \( Y, W \in X(M) \) implies that

\[
h(X, JX) = h(X, Z) = h(JX, Z) = 0
\]

\[
h(X, X) = h(JX, JX) = h(Z, Z) = H = \alpha JZ \tag{3.6}
\]

for some smooth function \( \alpha \) on \( M \), since \( H \in JD \).
Using (2.3) with $N = JZ$ we get
\[ AX = \alpha X, \quad AJX = \alpha JX, \quad AZ = \alpha Z \]  
(3.7)
So the frame field \{X, JX, Z\} diagonalizes $A$. Now using the equation $(\overline{\nabla}_X)(X) = 0$ and $(\overline{\nabla}_J)(X) = 0$ with the help of (3.6) we get
\[ g(\overline{\nabla}_X X, Z) = 0, \quad g(\overline{\nabla}_J X, Z) = 0 \]  
(3.8)
Also using the equation $\nabla_Z Z = 0$ from Lemma 1 we have
\[ g(\nabla_Z X, Z) = 0, \quad g(\nabla_Z JX, Z) = 0 \]  
(3.9)
Then using the equation $(\overline{\nabla}_X)(Z) = 0$ and (3.7) we obtain
\[ g(\overline{\nabla}_X Z, X) = 0, \quad g(\overline{\nabla}_X JX, X) = 0 \]  
(3.10)
and using the equation $(\overline{\nabla}_J)(Z) = 0$ we have
\[ g(\overline{\nabla}_J Z, X) = -\alpha, \quad g(\overline{\nabla}_J Z, JX) = 0 \]  
(3.11)
Using equations (3.8), (3.9), (3.10), and (3.11) one can write the following equations for the frame field \{X, JX, Z\}:
\[ \nabla_X Z = \alpha JX, \quad \nabla_J X Z = -\alpha X, \quad \nabla_Z Z = 0 \]  
(3.12)
\[ \nabla_X JX = aJX, \quad \nabla_J X JX = -bJX + \alpha Z, \quad \nabla_Z JX = cJX \]
for some smooth functions $a, b$ and $c$.

Now we are ready to prove the following:

**LEMMA 2.** For the frame field \{X, JX, Z\} we have
(i) \( R(X, Z; Z, X) = \|H\|^2 \)
(ii) \( R(X, JX; JX, X) = R(X, JX; JX, X) + \|H\|^2 \)
(iii) \( R(Z, JX, JX, Z) = \|H\|^2 \)

**PROOF.** Using equations (3.12) in the equation
\[ R(X, Z; Z, X) = g(\nabla_X \nabla_Z Z - \nabla_Z \nabla_X Z, -\nabla_X JX, X), \]
we obtain (i) and (iii). (ii) follows from the Gauss equation (2.6) and the equation $h(X, Y) = g(X, Y)H$.

**PROOF OF THE THEOREM.** Since $R(X, JX; JX, X) > 0$ and $\|H\| \neq 0$ it follows from (i), (ii), and (iii) of Lemma 2 that all plane sections of $M$ have strictly positive sectional curvature. Therefore, the Ricci-curvature of $M$ is strictly positive. Hence by Hamilton’s theorem (cf. [4]) it follows that $M$ is diffeomorphic to either $S^3, RP^3$ or the lens space $L^3_{p, q}$.

**ACKNOWLEDGEMENT.** The author would like to thank Dr. Sharief Deshmukh for useful discussion while preparing this paper.

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