APPROXIMATION BY FINITE RANK OPERATORS WITH RANGES IN c_0

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ABSTRACT. In this paper the author characterizes all those spaces X , for which $K_n(X, c_0)$ is proximal in $L(X, c_0)$. Some examples were found that satisfy this characterization.

KEY WORDS AND PHRASES. Proximal, best approximation, selection, extremal subspaces, n -width.

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1. INTRODUCTION.

The closed subset A of the normed linear space X , is said to be "*proximal*" in X if for each $x \in X$, there is an element $y_x \in A$, such that:

$$d(x, A) = \inf\{\|x - y\|; y \in A\} = \|x - y_x\|,$$

where $d(x, A)$ is the distance of x from A . The element y_x is called a "*best approximation*" of x in A . The best approximation need not be unique, and the set-valued function $P_A: X \rightarrow 2^A$ defined by

$$P_A = \{(y \in A; d(x, A) = \|x - y\|)\}$$

is called the *metric projection* of X into A . If A is proximal in X then $P_A(x) \neq \emptyset$ for each $x \in X$, in this case any function $f: X \rightarrow A$ satisfying that $f(x) \in P_A(x)$ for each $x \in X$, is called a "*selection*" for the metric projection P_A .

If A is a subset of X , and N is a subspace of X , then the "*deviation*" of A from N is defined to be

$$\delta(A, N) = \sup\{d(x, N); x \in A\},$$

and the n -width of A in X is defined to be

$$d_n(A, X) = \inf\{\delta(A, N); N \text{ is an } n\text{-dimensional subspace of } X\}.$$

If there is an n -dimensional subspace N_0 of X , such that $d_n(A, X) = \delta(A, N_0)$ then $d_n(A, X)$ is said to be "*attained*", and the subspace N_0 is said to be an "*extremal subspace*" for $d_n(A, X)$. It is well known (see Garkavi [4]), that if X^* is the dual space of the normed linear space X , then $d_n(A, X^*)$ is attained.

If X and Y are two normed linear spaces, then $L(X, Y)$ denotes the set of all bounded linear operators from X to Y , $K(X, Y)$ the set of all compact operators in $L(X, Y)$, and $K_n(X, Y)$ the subset of $K(X, Y)$ consisting of all operators of rank n .

The proximality of $K(X, Y)$ in $L(X, Y)$ were studied by several authors, (see for examples Feder [3], Lau [8], Mach [9], Mach and Ward [10], and Saatkamp [11]). Duetsch, Mach, and

Saatkamp [1], Kamal ([5], [6], and [7]) studied the proximality of $K_n(X, Y)$ in $L(X, Y)$ and $K(X, Y)$ in details, however, one of the problems left unsolved is the problem 5.2.2 mentioned by Duetsch. Mach and Saatkamp [1], concerning the proximality of $K_n(X, c_o)$, in $L(X, c_o)$ where c_o is the space of all real sequences that converges to zero. The problem is divided into two parts, the first part is to characterize all those spaces X for which $K_n(K, c_o)$ is proximal in $L(X, c_o)$, and the second part is to show whether $K_n(X, c_o)$ is proximal in $L(X, c_o)$ or not, when $X = c$ or l_∞ . Kamal [7] showed that $K_n(c, c_o)$ is not proximal in $L(c, c_o)$, given a partial solution for the second part of the mentioned problem. Deutsch, Mach, and Saatkamp [1] showed that if $X = c_o$ or if X^* is uniformly convex, then $K_n(X, c_o)$ is proximal in $L(X, c_o)$, Kamal [6] showed that $K_n(l_1, c_o)$ is not proximal in $L(l_1, c_o)$, also Kamal [7] showed that if Q is a compact Hausdorff space that contains an infinite convergent sequence, then $K_n(C(Q), c_o)$ is not proximal in $L(C(Q), c_o)$. In this paper a theorem is proved to characterize all those spaces X , for which $K_n(X, c_o)$ is proximal in $L(X, c_o)$, this characterization includes $X = c_o$, X for which X^* is uniformly convex, and X such that the metric projection P_N from X^* onto any of its n -dimensional subspaces N , has a selection which is ω^* -continuous at zero. A point worth mentioning is that although c_o is a one codimensional subspace of c , there are spaces X for which $K_n(X, c_o)$ is proximal in $L(X, c_o)$, meanwhile $K_n(X, c)$ is not proximal in $L(X, c)$, for example Deutsch, Mach and Saatkamp [1] showed that $K_n(c_o, c_o)$ is proximal in $L(c_o, c_o)$, meanwhile Kamal [7] showed that $K_n(c_o, c)$ is not proximal in $L(c_o, c)$.

The rest of introduction will cover some definitions, and known results that will be used later in Section 2.

If X is a normed linear space then $c_o(X^*, \omega^*)$ denotes the Banach space of all bounded sequences $\{x_i\}$ in X^* that converge to zero in the ω^* -topology induced on X^* by X , $c_o(X^*)$ is the Banach space of all sequences $\{x_i\}$ in X^* that converge to zero in the topology defined on X^* by its norm, and if $n \geq 1$ is any positive integer, then $c_o(X^*, n)$ denotes the union of all $c_o(N)$, where N is an n -dimensional subspace of X^* . The norm on $c_o(X^*, \omega^*)$ is the supremum norm. If $\{x_i\}$ is an element in $c_o(X^*, \omega^*)$ then for any positive integer $n \geq 1$, define

$$a_n(\{x_i\}) = \inf\{\|\{x_i\} - \{y_i\}\|; \{y_i\} \in c_o(X^*, n)\}$$

The following theorem can be obtained as a corollary, from the theorem of Dunford and Shwartz [2, p. 490].

THEOREM 1.1. Let X be normed linear space. The mapping $\alpha: L(X, c_o) \rightarrow c_o(X^*, \omega^*)$ defined by $\alpha(T)_i(x) = T(x)_i$ where $i = 1, 2, \dots$, and $x \in X$, is an isometric isomorphism. Furthermore $\alpha(K(X, c_o)) = c_o(X^*)$ and $\alpha(K_n(X, c_o)) = c_o(X^*, n)$.

As corollary of the Theorem 1.1, one can obtain the following:

COROLLARY 1.2. If X is a normed linear space then for any positive integer $n \geq 1$, the set $K_n(X, c_o)$ is proximal in $L(X, c_o)$ (resp. $K_n(X, c_o)$) if and only if $c_o(X^*, n)$ is proximal in $c_o(X^*, \omega^*)$ (resp. $c_o(X^*)$).

According to Corollary 1.2 to study the proximality of $K_n(X, c_o)$ in $L(X, c_o)$ (resp. $K(X, c_o)$), it is enough to study the proximality of $c_o(X^*, n)$ in $c_o(X^*, \omega^*)$ (resp. $c_o(X^*)$).

2. THE PROXIMALITY OF $K_n(X, c_o)$ IN $L(X, c_o)$.

In this paper if $\{x_i\}$ is an element in $c_o(X^*, \omega^*)$, then $d_n(\{x_i\}, X^*)$ (resp. $\delta(\{x_i\}, N)$) denotes the n -width (resp. the deviation from N) of the subset $\{x_1, x_2, x_3, \dots\}$ of X^* .

THEOREM 2.1 Let X be a normed linear space, and let $n \geq 1$ be any positive integer. If $\{x_i\}$ is a bounded sequence in X^* then

$$a_n(\{x_i\}) = \max\{d_n(\{x_i\}, X^*), \overline{\lim} \|x_i\|\}.$$

Furthermore there is an n -dimensional subspace N_o of X^* , such that $a_n(\{x_i\}) = d(\{x_i\}, c_o(N_o))$.

PROOF. First it will be shown that $a_n(\{x_i\}) \geq \max\{d_n(\{x_i\}, N^*), \overline{\lim} \|x_i\|\}$. By Garkavi [4], there is an n -dimensional subspace N_o of X^* such that $\delta(\{x_i\}, N) = d_n(\{x_i\}, X^*)$. For each $i = 1, 2, \dots$, let z_i be a best approximation for x_i from N_o , and let $\varepsilon > 0$ be given, there is a positive integer $n \geq 1$ such that for each $i \geq m$, $\|x_i\| \leq \overline{\lim} \|x_i\| + \varepsilon$. Define the sequence $\{y_i\}$ in $c_o(N_o)$ as follows.

$$y_i = \begin{cases} z_i & \text{if } i \leq m \\ 0 & \text{if } i > m. \end{cases}$$

Then

$$\begin{aligned} a_n(\{x_i\}) &\leq \|\{x_i\} - \{y_i\}\| = \sup\{\|x_i - y_i\|; i = 1, 2, \dots\} \\ &= \max\{\max\{\|x_i - y_i\|; i = 1, 2, \dots, m\}, \sup\{\|x_i\|; i = m+1, m+2, \dots\}\} \\ &\leq \max\{d_n(\{x_i\}, X^*), \overline{\lim} \|x_i\| + \varepsilon\}. \end{aligned}$$

Since ε is arbitrary it follows that $a_n(\{x_i\}) \leq \max\{d_n(\{x_i\}, X^*), \overline{\lim} \|x_i\|\}$. Second to show that $a_n(\{x_i\}) \geq \max\{d_n(\{x_i\}, X^*), \overline{\lim} \|x_i\|\}$, one should notice first that $a_n(\{x_i\}) \geq \overline{\lim} \|x_i\|$, indeed if $\{y_i\} \in c_o(X^*, n)$ then

$$\|\{x_i\} - \{y_i\}\| = \sup\{\|x_i - y_i\|\} \geq \overline{\lim} \|x_i - y_i\| = \overline{\lim} \|x_i\|.$$

Let $\varepsilon > 0$ be given, there is an n -dimensional subspace N of X^* , and a sequence $\{y_i\} \in c_o(N)$ such that $a_n(\{x_i\}) \geq \|\{x_i\} - \{y_i\}\| - \varepsilon$. Therefore

$$\|\{x_i\} - \{y_i\}\| = \sup\{\|x_i - y_i\|\} \geq \sup d(x_i, N) = \delta(\{x_i\}, N) \geq d_n(\{x_i\}, X^*).$$

Hence $a_n(\{x_i\}) = d(\{x_i\}, X^*) - \varepsilon$, and since ε is arbitrary it follows that

$$a_n(\{x_i\}) \geq d_n(\{x_i\}, X^*).$$

To prove the fact that there is an n -dimensional subspace N of X^* , such that $a_n(\{x_i\}) = d_n(\{x_i\}, c_o(N))$, Let N be an extremal subspace for $d_n(\{x_i\}, X^*)$, and for each $i = 1, 2, \dots$, let z_i be a best approximation for x_i from N . Let $\varepsilon > 0$ be given, and define the sequence $\{y_i\}$ in $c_o(N)$ as in the first part of the proof, then

$$\begin{aligned} \|\{x_i\} - \{y_i\}\| &= \sup\{\|x_i - y_i\|\} \\ &= \max\{\max\{\|x_i - z_i\|; i = 1, 2, \dots, m\}, \sup\{\|x_i\|; i = m+1, m+2, \dots\}\} \\ &\leq \max\{\delta(\{x_i\}, N), \overline{\lim} \|x_i\| + \varepsilon\} \\ &\leq \max\{d_n(\{x_i\}, X^*), \overline{\lim} \|x_i\| + \varepsilon\} \\ &= a_n(\{x_i\}) + \varepsilon. \end{aligned}$$

But ε is arbitrary so $d(\{x_i\}, c_o(N)) = a_n(\{x_i\})$.

THEOREM 2.2. Let X be a normed linear space. For any positive integer $n \geq 1$, $K_n(X, c_o)$ is proximal in $K(X, c_o)$.

PROOF. Let $\{x_i\}$ be an element in $c_o(X^*)$, by Corollary 1.2, it is enough to find an element $\{y_i\}$ in $c_o(X^*, n)$ such that $\|\{x_i\} - \{y_i\}\| = a_n(\{x_i\})$. Since $\lim_{i \rightarrow \infty} \|x_i\| = 0$ it follows that $\overline{\lim} \|x_i\| = 0$, thus by Theorem 2.1, $a_n(\{x_i\}) = d_n(\{x_i\}, X^*)$. Let N_o be an extremal subspace for $d_n(\{x_i\}, X^*)$, and for each $i = 1, 2, \dots$, let y_i be a best approximation for x_i from N_o . Since $\lim_{i \rightarrow \infty} \|x_i\| = 0$, it follows that $\lim_{i \rightarrow \infty} \|y_i\| = 0$; that is, $\{y_i\} \in c_o(N_o)$. Thus

$$\|\{x_i\} - \{y_i\}\| = \sup\{\|x_i - y_i\|\} = \delta(\{x_i\}, N_o) = d_n(\{x_i\}, X^*) = a_n(\{x_i\}).$$

LEMMA 2.3. Let X be a normed linear space, and let $\{x_i\}$ be a bounded sequence in X^* .

a) If $d_n(\{x_i\}, X^*) > \overline{\lim} \|x_i\|$, then $a_n(\{x_i\})$ is attained.

b) If $d_n(\{x_i\}, X^*) \leq \overline{\lim} \|x_i\|$, and there is an extremal subspace N_o for $d_n(\{x_i\}, X^*)$ such that $\overline{\lim} d(x_i, N_o) < \overline{\lim} \|x_i\|$, then $a_n(\{x_i\})$ is attained.

PROOF. a) Assume that N is an extremal subspace for $d_n(\{x_i\}, X^*)$, and let $\alpha = d_n(\{x_i\}, X^*) - \overline{\lim} \|x_i\|$, then there is a positive integer $m \geq 1$ such that for each $i \geq m$, one has $\|x_i\| \leq \overline{\lim} \|x_i\| + \alpha$. For each $i \leq m$, let z_i be a best approximation for x_i from N_o , and define the sequence $\{y_i\}$ in $c_o(N_o)$ as follows.

$$y_i = \begin{cases} z_i & \text{if } i \leq m \\ 0 & \text{if } i > m. \end{cases}$$

Then

$$\begin{aligned} \|\{x_i\} - \{y_i\}\| &= \max\{\max\{\|x_i - z_i\|; i = 1, 2, \dots, m\}, \sup\{\|x_i\|; i = m+1, m+2, \dots\}\} \\ &\leq \max\{\delta(\{x_i\}, N_o), \overline{\lim} \|x_i\| + \alpha\} \\ &= d_n(\{x_i\}, X^*) = a_n(\{x_i\}). \end{aligned}$$

b) let $\alpha = \overline{\lim} \|x_i\|$, $\beta = \overline{\lim} d(x_i, N_o)$, and $\gamma = \alpha - \beta$. Then $\gamma > 0$.

Let $\{\varepsilon_i\}$ be a sequence of positive real numbers, satisfying that $\lim_{i \rightarrow \infty} \varepsilon_i = 0$, for each $i = 1, 2, \dots$, $d(x_i, N_o) \leq \beta + \varepsilon_i$ and for each $i = 1, 2, \dots$, $\|x_i\| \leq \alpha + \varepsilon_i$. For each $i = 1, 2, \dots$, let z_i be a best approximation for x_i from N_o , and define the sequence $\{y_i\}$ in N_o as follows,

$$y_i + \begin{cases} z_i & \text{if } \varepsilon_i \geq \gamma \\ \frac{\varepsilon_i}{\gamma} z_i & \text{if } \varepsilon_i < \gamma. \end{cases}$$

Since $\{z_i\}$ is a bounded sequence in N_o , and $\lim_{i \rightarrow \infty} \varepsilon_i = 0$, it follows that $\{y_i\} \in c_o(N_o)$.

Furthermore for each $i = 1, 2, \dots$, if $\varepsilon_i > \gamma$ then

$$\|x_i - y_i\| \leq d(x_i, N_o) \leq d_n(\{x_i\}, X^*) \leq a(\{x_i\}),$$

and if $\varepsilon_i < \gamma$ then

$$\begin{aligned} \|x_i - y_i\| &\leq (1 - \frac{\varepsilon_i}{\gamma}) \|x_i\| + \frac{\varepsilon_i}{\gamma} \|x_i - z_i\| \\ &\leq (1 - \frac{\varepsilon_i}{\gamma})(\alpha + \varepsilon_i) + \frac{\varepsilon_i}{\gamma}(\alpha - \gamma + \varepsilon_i) \\ &= \alpha = a_n(\{x_i\}). \end{aligned}$$

Thus $\|\{x_i\} - \{y_i\}\| = a_n(\{x_i\})$.

Lemma 2.4 is a continuation for Lemma 2.3.

LEMMA 2.4. Let X be a normed space, and let $\{x_i\}$ be a bounded sequence in X^* . Assume that $d_n(\{x_i\}, X^*) = \overline{\lim} \|x_i\|$, and for each extremal subspace N for $d_n(\{x_i\}, X^*)$ one has $\overline{\lim} d(x_i, N) = \overline{\lim} \|x_i\| = \alpha$. Let N be an extremal subspace for $d_n(\{x_i\}, X^*)$, and for each $i = 1, 2, \dots$, define

$$\varepsilon_i = \begin{cases} 0 & \text{if } \|x_i\| \leq \alpha \\ \|x_i\| - \alpha & \text{if } \|x_i\| > \alpha \end{cases}, \quad \delta_i = \alpha - d(x_i, N_o), \text{ and } \alpha_i = \begin{cases} 0 & \text{if } \varepsilon_i + \delta_i = 0 \\ \frac{\varepsilon_i}{\varepsilon_i + \delta_i} & \text{if } \varepsilon_i + \delta_i \neq 0. \end{cases}$$

If $\lim_{i \rightarrow \infty} \alpha_i = 0$ then $a_n(\{x_i\})$ is attained.

PROOF. Let z_i be a best approximation for x_i from N_o , and let $y_i = \alpha_i z_i$, then the sequence $\{y_i\}$ is an element in $c_o(N_o)$. Furthermore for each $i = 1, 2, \dots$,

$$\begin{aligned} \|x_i - y_i\| &\leq (1 - \alpha_i) \|x_i\| + \alpha_i \|x_i - z_i\| \\ &\leq (1 - \alpha_i)(\alpha + \varepsilon_i) + \alpha_i(\alpha - \delta_i) \\ &= \alpha + \varepsilon_i - \alpha_i(\varepsilon_i + \delta_i). \end{aligned}$$

If $\alpha_i = 0$ then $\varepsilon_i = 0$ so $\|x_i - y_i\| = \alpha$, and if $\alpha_i \neq 0$ then

$$\|x_i - y_i\| \leq \alpha + \varepsilon_i - \frac{\varepsilon_i}{\varepsilon_i + \delta_i}(\varepsilon_i + \delta_i) = \alpha.$$

DEFINITION 2.5. Let X be a normed linear space. The bounded sequence $\{x_i\}$ in $c_0(X^*, \omega^*)$ is said to be an " n -border" sequence if it satisfies the following,

1. $\lim_{i \rightarrow \infty} \|x_i\|$ exists, and for each extremal subspace N for $d_n(\{x_i\}, X^*)$, one has

$$\overline{\lim} d(x_i, N) = \lim_{i \rightarrow \infty} \|x_i\| = d_n(\{x_i\}, X^*).$$

2. For each extremal subspace N for $d_n(\{x_i\}, X^*)$ if ε_i , δ_i and α_i as in Lemma 2.4 then $\overline{\lim} \alpha_i > 0$.

THEOREM 2.6. Let X be a normed linear space, and let $n \geq 1$ be a positive integer. Then $K_n(X, c_0)$ is proximal in $L(X, c_0)$ if and only if for each n -border sequence $\{x_i\}$ in X^* , $a_n(\{x_i\})$ is attained.

PROOF. If there is an n -border sequence $\{x_i\}$ in X^* such that $a_n(\{x_i\})$ is not attained, then since $\{x_i\} \in c_0(X^*, \omega^*)$, it follows by Corollary 1.2 that $K_n(X, c_0)$ is not proximal in $L(X, c_0)$. To prove the other part, let $\{x_i\}$ be an element in $c_0(X^*, \omega^*)$. If $d_n(\{x_i\}, X^*) > \overline{\lim} \|x_i\|$, or if $d_n(\{x_i\}, X^*) \leq \overline{\lim} \|x_i\|$ and there is an extremal subspace N for $d_n(\{x_i\}, X^*)$, such that $\lim d(x_i, N) < \overline{\lim} \|x_i\|$ then by Lemma 2.3, $a_n(\{x_i\})$ is attained.

Assume that $d_n(\{x_i\}, X^*) = \overline{\lim} \|x_i\|$, and for each extremal subspace N for $d_n(\{x_i\}, X^*)$, one has $\overline{\lim} d(x_i, N) = \overline{\lim} \|x_i\|$, let ε_i , δ_i and α_i be as in Lemma 2.4. If there is an extremal subspace N for $d_n(\{x_i\}, X^*)$ such that $\lim_{i \rightarrow \infty} \alpha_i = 0$ then by Lemma 2.4, $a_n(\{x_i\})$ is attained. Therefore one may assume that for any extremal subspace N for $d_n(\{x_i\}, X^*)$ one has $\overline{\lim} \alpha_i > 0$. Let $\alpha = d_n(\{x_i\}, X^*)$ and let $\{x_{i_k}\}$ be the largest subsequence of $\{x_i\}$ satisfying that $\|x_{i_k}\| > \alpha$ for each i_k . Thus for each i , if x_i is not an element in $\{x_{i_k}\}$ then $\|x_i\| \leq \alpha$. The sequence $\{x_{i_k}\}$ is an n -border sequence in X^* , so there is an n -dimensional subspace N of X^* , and a sequence $\{z_{i_k}\} \in c_0(N)$ such that $\|x_{i_k}\| - \|z_{i_k}\| = a_n(\{x_{i_k}\}) = \alpha$.

Define the sequence $\{y_i\}$ in N as follows.

$$y_i = \begin{cases} z_{i_k} & \text{if } \|x_i\| > \alpha \\ 0 & \text{if } \|x_i\| \leq \alpha. \end{cases}$$

Then $\{y_i\} \in c_0(N)$, and $\|\{x_i\} - \{y_i\}\| = \alpha = a_n(\{x_i\})$.

COROLLARY 2.7. Let X be a normed linear space, and let $n \geq 1$ be a positive integer. If X^* is uniformly convex then $K_n(X, c_0)$ is proximal in $L(X, c_0)$.

PROOF. Let $\{x_i\}$ be an n -border sequence in X^* , and let $\alpha = \lim_{i \rightarrow \infty} \|x_i\|$. Without loss of generality assume that $x_i \neq 0$ for each i . Let N be any extremal subspace for $d_n(\{x_i\}, X^*)$, and let y_i be the best approximation for x_i from N . Since $\|\frac{x_i}{\|x_i\|}\| = 1$, $\frac{\|x_i - y_i\|}{\alpha} \leq 1$, and

$$\lim_{i \rightarrow \infty} \left\| \frac{x_i}{\|x_i\|} + \frac{x_i - y_i}{\alpha} \right\| = \lim_{i \rightarrow \infty} \left(\frac{\alpha + \|x_i\|}{\alpha \|x_i\|} \right) \|x_i - y_i\| \geq \lim_{i \rightarrow \infty} \left(\frac{\alpha + \|x_i\|}{\alpha \|x_i\|} \right) \|x_i - y_i\| = 2.$$

It follows by the fact that X^* is uniformly convex that $\lim_{i \rightarrow \infty} \left\| \frac{x_i}{\|x_i\|} - \frac{x_i - y_i}{\alpha} \right\| = 0$. But then $\lim_{i \rightarrow \infty} y_i = 0$, so $\{y_i\} \in c_0(N)$ and $\|\{x_i\} - \{y_i\}\| = a_n(\{x_i\})$.

Corollary 2.7 was proved by Deutsch, Mach, and Saatkamp [1] in a different way.

COROLLARY 2.8. Let X be a normed linear space, and let $n \geq 1$ be a positive integer. If for each n -dimensional subspace N of X^* , the metric projection P_N has a selection which is ω^* -continuous at zero, then $K_n(X, c_0)$ is proximal in $L(X, c_0)$.

PROOF. Let $\{x_i\}$ be an element in $c_0(X^*, \omega^*)$ and let N be an extremal subspace for $d_n(\{x_i\}, X^*)$. Since the metric projection P_N has a selection which is ω^* -continuous at zero, it follows that there is a sequence $\{y_i\}$ in N , satisfying that $y_i \in P_N(x_i)$ for each i , and that $\{y_i\}$

converges ω^* -to zero. But N is of finite dimension, thus $\{y_i\} \in c_o(N)$.

Furthermore

$$\|\{x_i\} - \{y_i\}\| = \delta(\{x_i\}, N) = d_n(\{x_i\}, X^*) = a_n(\{x_i\}).$$

From Corollary 2.8 one concludes that for each positive integer $n \geq 1$, if $X = c_o$ or $l_p, 1 < p < \infty$, then $K_n(X, c_o)$ is proximal in $L(X, c_o)$. Proposition 2.9 clarify that. The fact that $K_n(c_o, c_o)$ is proximal in $L(c_o, c_o)$ was proved first by Deutsch, Mach, and Saatkamp [1].

PROPOSITION 2.9. Let $n \geq 1$ be a positive integer and let $X = c_o$ or $l_p, 1 < p < \infty$. The metric projection P_N from X^* onto any of its n -dimensional subspace N , has a selection which is ω^* -continuous at zero.

PROOF. Let N be any n -dimensional subspace of X^* , $\{x_i\}$ be any bounded sequence in X^* that converges ω^* -to zero, and let $\{y_i\}$ be any sequence in N , satisfying that $y_i \in P_N(x_i)$ for each i . It will be shown that $\{y_i\} \in c_o(N)$. The sequence $\{y_i\}$ is a bounded sequence in a finite dimensional subspace of X^* , so it has a convergent subsequence $\{y_{i_k}\}$ that converges to y_o in N , it will be shown that $y_o = 0$. Assume not, and without loss of generality assume that $\{y_i\}$ converges to y_o , and that $X^* = l_p, 1 \leq p < \infty$. Let $t_i = x_i - (y_i - y_o)$, $r_i = x_i - y_i$, and let $\varepsilon > 0$ be such that $\varepsilon < \|y_o\|^p$, then as in Proposition 3 of Mach [9], there is a positive integer $m \geq 1$ such that for each $i \geq m$ one has, $|\|t_i - y_o\|^p - \|t_i\|^p - \|y_o\|^p| < \varepsilon$, thus $\|t_i - y_o\|^p \geq \|t_i\|^p + \|y_o\|^p - \varepsilon$, that is

$$\|x_i - y_i\|^p \geq \|x_i - (y_i - y_o)\|^p + \|y_o\|^p - \varepsilon > \|x_i - (y_i - y_o)\|^p.$$

So for each $i > m$ one has $\|x_i - (y_i - y_o)\| > \|x_i - y_i\|$, which contradict the fact that $\|x_i - y_i\| = d(x_i, N)$, therefore $y_o = 0$.

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