COMPLETELY POSITIVE LINEAR OPERATORS FOR BANACH SPACES

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ABSTRACT. Using ideas of Pisier, the concept of complete positivity is generalized in a different direction in this paper, where the Hilbert space \( \mathcal{H} \) is replaced with a Banach space and its conjugate linear dual. The extreme point results of Arveson are reformulated in this more general setting.

KEY WORDS AND PHRASES: Banach spaces, completely positive operators, extreme points, pure elements.

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1. INTRODUCTION.

In [6], Pisier studied completely bounded maps from a \( C^* \)-algebra to \( B(X,Y) \), the space of bounded operators between two arbitrary Banach spaces \( X \) and \( Y \). Of course, there is a generalization of ordinary completely bounded maps. In this paper, we first define complete positivity for a map from \( C^* \)-algebra to \( B(X,X^\ast) \), where \( X^\ast \) denotes the antilinear dual space of \( X \) (the set of all conjugate linear functionals on \( X \)). Then we give a representation theorem, and give complete solutions to three extremal problems.

In this paper, the \( C^* \)-algebra \( A \) always has an identity.

2. COMPLETELY POSITIVE OPERATORS.

DEFINITION 2.1. Let \( X \) be a Banach space, and \( T \in B(X,X^\ast) \). We call \( T \) positive if, for all positive integers \( n \) and \( x_1,\ldots,x_n \in X \), we have

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} T(x_i)(x_j) \geq 0.
\]

REMARK 2.2. We have \( T(X)^* = T(X)^\ast \), and so \( M_n(B(X,X^\ast)) = B((\ell_2^n(X))^*,(\ell_2^n(X))^\ast) = B((\ell_2^n(X)),(\ell_2^n(X))^\ast) \). Thus we may define positivity for \( M_n(B(X,X^\ast)) \).

DEFINITION 2.3. Let \( A \) be a \( C^* \)-algebra, \( \phi \) a linear map from \( A \) to \( B(X,X^\ast) \) and let \( \phi_n(a_{ij}) = (\phi(a_{ij})) \) for \( (a_{ij}) \in M_n(A) \). If \( \phi \) is positive for all \( n \), then we say \( \phi \) is completely positive.

THEOREM 2.4. Let \( \phi : A \rightarrow B(X,X^\ast) \) be a completely positive map. There is a Hilbert space \( \mathcal{H} \), a representation \( \pi \) of \( A \) on \( \mathcal{H} \) and a bounded operator \( V \in B(X,\mathcal{H}) \) such that, for all \( a \in A \),

\[
\phi(a) = \overline{V}^* \pi(a) V,
\]

and \( \mathcal{H} = [\pi(A) V X] \), where \( \overline{V}^*(h)(x) = \langle h, V(x) \rangle \), for all \( h \in \mathcal{H}, x \in X \).
PROOF: Consider the vector space tensor product $A \otimes X$ and define a bilinear form as follows:

If $u = x_1 \otimes \xi_1 + \ldots + x_m \otimes \xi_m$, $v = y_1 \otimes \eta_1 + \ldots + y_n \otimes \eta_n$,

$$< u, v > = \sum_{i,j} (\phi(y_j^* x_j)(\xi_j))(\eta_j).$$

Because $\phi$ is completely positive, we have the fact that $<,>$ is positive semi-definite. For each $a \in A$, define a linear transformation $\pi_0(a)$ on $A \otimes X$ by

$$\pi_0(a)(\sum_{j=1}^n x_j \otimes \xi_j) = \sum (ax_j) \otimes \xi_j.$$ 

$\pi_0$ is an algebra homomorphism for which

$$< u, \pi_0(a)v > = < \pi_0(a^*u), v >$$

for all $u, v \in A \otimes X$.

For fixed $u$, $\rho(a) = < \pi_0(a)u, u >$ defines a positive linear functional on $A$; i.e., $\rho(a^*a) \geq 0$. Hence, $< \pi_0(a)u, \pi_0(a)u > = < \pi_0(a^*a)u, u > = \rho(a^*a) \leq \|a^*a\|\rho(1) = \|a\|^2 < u, u >$, where 1 is the identity of $A$.

Now let $R = \{ u \in A \otimes X : < u, u > = 0 \}$. $R$ is a linear subspace $A \otimes X$, invariant under $\pi_0(a)$, for all $a \in A$. So $<,>$ determines a positive definite inner product on the quotient $(A \otimes X)/R$ in the usual way.

Let $\mathcal{H} = (A \otimes X)/R$. There is a unique representation $\pi$ of $A$ on $\mathcal{H}$ such that

$$\pi(a)(u + R) = \pi_0(a)u + R$$

$a \in A, u \in A \otimes X$.

We define a linear map $V: X \rightarrow \mathcal{H}$ by

$$V(\xi) = 1 \otimes \xi + R$$

for all $\xi \in X$.

We may verify that $V$ is bounded, and $\phi(a) = \overline{V^*} \pi(a)V$ for all $a \in A$.

Let $R_1 = [\pi(A)VX] \subseteq \mathcal{H}$, and $\pi_1(a) = \pi(a)|_{R_1}$ for all $a \in A$. Because $\pi(1) = I$, so $V(X) \subseteq R_1$. We have $\overline{V^*} \pi_1(a)V(x_1) = \overline{V^*} \pi_1(a)|_{R_1}V(x_1) = \overline{V^*} \pi_1(a)V_1(x_1) = \phi(a)(x_1)$, for all $x_1 \in X, a \in A$. So we may assume that $\mathcal{H} = [\pi(A)VX]$.

Suppose $\phi: A \rightarrow B(X, X^*)$ is a completely positive map. If there exists Hilbert spaces $\mathcal{H}_i$, representations $\pi_i$ of $A$ on $\mathcal{H}_i$, and bounded operators $V_i: X \rightarrow \mathcal{H}_i$ then

$$\phi(a) = \overline{V_i^*} \pi_i(a)V_i,$$

for $i = 1, 2$, where $\mathcal{H}_i = [\pi_i(A)V_iX]$. Define $U: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ by

$$U(\sum_{i=1}^n \pi_1(a_i)V_1 \xi_i) = \sum_{i=1}^n \pi_2(a_i)V_2 \xi_i,$$

for all $a_1, \ldots, a_n \in A, \xi_1, \ldots, \xi_n \in X$. Then we need to extend to $\mathcal{H}_1$. We may verify that $U V_1 = V_2$ and $U \pi_1(a) = \pi_2(a)U$ for all $a \in A$. 

Next we verify that \( U \) is an unitary.

\[
\begin{align*}
\langle \sum_{i=1}^{n} \pi_2(a_i)V_2\xi_i, \sum_{i=1}^{n} \pi_2(a_i)V_2\xi_i \rangle &= \sum_{i,j=1}^{n} \langle \pi_2(a_i)V_2\xi_i, \pi_2(a_j)V_2\xi_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \pi_2(a_i^*a_i)V_2\xi_i, V_2\xi_j \rangle \\
&= \sum_{i,j=1}^{n} \phi(a_i^*a_i)(\xi_i)(\xi_j) \\
&= \sum_{i,j=1}^{n} \langle \pi_1(a_i^*a_i)V_1\xi_i, V_1\xi_j \rangle \\
&= \langle \sum_{i=1}^{n} \pi_1(a_i)V_1\xi_i, \sum_{i=1}^{n} \pi_1(a_i)V_1\xi_i \rangle .
\end{align*}
\]

So the representation given in Theorem 2.4 is unique up to unitary equivalence.

3. PREPARATIONS.

**NOTATION 3.1.** Let \( CP(A, X) \) denote all completely positive linear maps from \( A \) to \( B(X, X^*) \).

**LEMMA 3.2.** Let \( \phi_1 \) and \( \phi_2 \) belong to \( CP(A, X) \), and suppose that \( \phi_1 \preceq \phi_2 \). Let \( \phi_i(a) = \sum_{j=1}^{n} \pi_i(a)V_j \) be the canonical expression of \( \phi_i \), where \( \pi_i \) is a representation of \( A \) on \( R_i \) such that \( [\pi_i(A)V_j, V_k] = 0, i = 1, 2 \). Then there exists a contraction \( T \in B(R_2, R_1) \) such that

\[
TV_2 = V_1,
\]

\[
T\pi_2(a) = \pi_1(a)T
\]

for all \( a \in A \).

**PROOF:** For every \( f, \ldots, f, x, a, \ldots, a, A \),

\[
\| \sum_{j=1}^{n} \pi_1(a_j)V_j\xi_j \|^2 = \sum_{j=1}^{n} \pi_1(a_j)V_j\xi_j, \sum_{j=1}^{n} \pi_1(a_j)V_j\xi_j
\]

\[
= \sum_{i,j} \pi_1(a_i^*a_i)V_1(\xi_i)(\xi_j)
\]

\[
= \sum_{i,j} \phi_1(a_i^*a_i)(\xi_i)(\xi_j)
\]

\[
\leq \sum_{i,j} \phi_2(a_i^*a_i)(\xi_i)(\xi_j)
\]

\[
= \| \sum_{j=1}^{n} \pi_2(a_j)V_2\xi_j \|^2
\]

Define \( T : R_2 \rightarrow R_1 \) by

\[
T(\sum_{j=1}^{n} \pi_2(a_j)V_2\xi_j) = \sum_{j=1}^{n} \pi_1(a_j)V_1\xi_j
\]

We can verify that above two statements hold.

**NOTATION 3.3.** For \( \phi \in CP(A, X) \), let \( [0, \phi] = \{ \psi \in CP(A, X); \psi \preceq \phi \} \). Let \( \phi(a) = \sum_{j=1}^{n} \pi(a)V_j \) for all \( a \in A \). For each operator \( T \in \pi(A)' \), define a map \( \phi_T(a) = \sum_{j=1}^{n} T\pi(a)V_j \). Then \( T \rightarrow \phi_T \) is linear. If \( \phi_T = 0 \), we have

\[
\langle T\pi(a)V_1\xi_j, \pi(b)V_1\eta \rangle = \langle T\pi(b^*a)V_1\xi_j, V_1\eta \rangle = \phi_T(b^*a)(\xi_j)(\eta) = 0
\]

\[
\langle \sum_{i=1}^{n} \pi_1(a_i)V_1\xi_j, \sum_{i=1}^{n} \pi_1(b_j)V_1\xi_j \rangle = 0.
\]

So \( T = 0 \). That is, \( T \rightarrow \phi_T \) is injective.
THEOREM 3.4. $T \rightarrow \phi_T$ is an affine order isomorphism of the partially ordered convex set of $\{T \in \pi(A)' : 0 \leq T \leq I\}$ onto $[0, \phi]$.

The proof of this theorem is exactly the same way as the proof of theorem in Arveson's paper [1].

4. THE THREE EXTREMAL PROBLEMS.

Now we come to discuss three extremal problems.

DEFINITION 4.1. A completely positive map $\phi \in CP(A, X)$ is pure if, for every $\psi \in CP(A, X)$, $\psi \leq \phi$ implies that $\psi$ is a scalar multiple of $\phi$.

REMARK 4.2. According to [3], the extreme rays of $CP(A, X)$ can be characterized as the half lines $\{t\phi : t \geq 0\}$, where $\phi$ is a pure element of $CP(A, X)$.

We state the following theorems without proofs, for the proofs are almost the same as those in Arveson's paper [1].

THEOREM 4.3. All nonzero pure elements of $CP(A, X)$ are precisely those of the form $\phi(a) = V^*\pi(a)V$, where $\pi$ is an irreducible representation of $A$ on some Hilbert space $R$ and $V \in B(X, R)$, such that $R = [\pi(A)VX]$.

THEOREM 4.4. Let $\phi \in CP(A, X)$ and let $\phi(a) = V^*\pi(a)V$ be its canonical representation. The extreme points of $[0, \phi]$ are those maps of the form $V^*P\pi(a)V$, where $P$ is a projection in $\pi(A)'$.

We consider the extreme points of the set $CP(A, X; K) = \{\phi \in CP(A, X) ; \phi(1) = K\}$, where $K$ is a fixed positive operator in $B(X, X^*)$.

THEOREM 4.5. Let $\phi \in CP(A, X; K)$ and let $\pi(a) = V^*\pi(a)V$ be its canonical representation with $V^*V = K$. Then $\phi$ is an extreme point of $CP(A, X; K)$ if and only if $[VX]$ is a faithful subspace for the commutant $\pi(A)'$ of $\pi(A)$.

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References

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