CR-SUBMANIFOLDS OF A LOCALLY CONFORMAL KAEHLER SPACE FORM

M. HASAN SHAHID

Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia, New Delhi - 110 025

(Received November 14, 1991 and in revised form May 20, 1992)

ABSTRACT. (Bejancu [1,2]) The purpose of this paper is to continue the study of CR -submanifolds, and in particular of those of a locally conformal Kaehler space form (Matsumoto [3]). Some results on the holomorphic sectional curvature, D -totally geodesic, D^1 -totally geodesic and D^1 -minimal CR -submanifolds of locally conformal Kaehler (l.c.k.)-space from $\bar{M}(c)$ are obtained. We have also discussed Ricci curvature as well as scalar curvature of CR -submanifolds of $\bar{M}(c)$.

KEY WORDS AND PHRASES. CR -submanifolds, D -totally geodesic, D^1 -totally geodesic and minimal CR -submanifolds.

1991 AMS SUBJECT CLASSIFICATION CODE. 53B25.

1. PRELIMINARIES.

Let $\bar{M}$ be a Hermitian manifold with complex structure J . Let Ω denote the fundamental 2-form of a Hermitian manifold $\bar{M}$ defined by $g(JX, Y) = \Omega(X, Y)$, where g is the Hermitian metric and X, Y are arbitrary vector fields on $\bar{M}$. $\bar{M}$ is called a locally conformal Kaehler (l.c.k.) manifold [4] if there is a closed 1-form called the Lee form on $\bar{M}$ such that $d\Omega = \Omega \wedge \omega$ where d and $\wedge$ denoting exterior derivative operator and wedge product. In a l.c.k. manifold $\bar{M}$, we define a symmetric tensor field $P(X, Y)$ as

$$P(Y, X) = -(\bar{\nabla}_Y \alpha)(X) - \alpha(X)\alpha(Y) + \frac{1}{2} \|\alpha\|^2 g(X, Y), \quad (1.1)$$

where $\|\alpha\|$ denotes the length of the Lee form with respect to g . Moreover, we assume that the tensor field P is hybrid, that is,

$$P(Y, JX) = -P(JY, X) \quad (1.2)$$

A l.c.k. manifold $\bar{M}$ is called a l.c.k.-space form if it has a constant holomorphic sectional curvature c , and will be denoted by $\bar{M}(c)$. Let $\bar{M}(c)$ be a l.c.k. - space form, and M be a Riemannian manifold isometrically immersed in $\bar{M}$. We denote by g the metric tensor field of $\bar{M}(c)$ as well as that induced on M . Let $\bar{\nabla}$ (resp. ∇) be the covariant differentiation with respect to the Levi-Civita connection in $\bar{M}$ (resp. M). Then the Gauss and Weingarten formulas for M are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_X N + \nabla_X^\perp N, \quad (1.3)$$

for any $X, Y \in TM$, where h (resp. A) is the second fundamental form (resp. tensor) of M and $\nabla^\perp$ denotes the operator of the normal connection. Moreover

$$g(h(X, Y), N) = g(A_N X, Y). \quad (1.4)$$

The curvature tensor $\bar{R}$ of a l.c.k. space form $\bar{M}(c)$ is given by Matsumoto [3]

$$\begin{aligned}\bar{R}(X, Y, Z, W) = & \frac{c}{4} [g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) \\ & - 2g(JX, Y)g(JZ, X)] + \frac{3}{4} [P(X, W)g(Y, Z) - P(X, Z)g(Y, W) + g(X, W)P(Y, Z) \\ & - g(X, Z)P(Y, W)] + \frac{1}{4} [P(X, JW)g(JY, Z) - P(X, JZ)g(JY, W) + g(JX, W)P(Y, JZ) \\ & - g(JX, Z)P(Y, JW) - 2P(X, JY)g(JZ, W) - 2P(Z, JW)g(JX, Y)],\end{aligned}\quad (1.5)$$

where $\bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W)$ and

$$P(X, Y) = P(Y, X), \quad P(X, JY) = -P(JX, Y), \quad P(JX, JY) = P(X, Y).$$

The Gauss equation is given by

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \quad (1.6)$$

where R (resp. $\bar{R}$) is the curvature of M and (resp. $\bar{M}(c)$).

DEFINITION 1.1. A submanifold M of a l.c.k. space form $\bar{M}(c)$ is called a CR -submanifold if there exists a differentiable distribution $D: x \rightarrow D_x \subset T_x M$ on M satisfying the following condition:

- (i) D is holomorphic i.e. $JD_x = D_x$ for each $x \in M$ and
- (ii) the complementary orthogonal distribution $D^\perp: x \rightarrow D_x^\perp \subset T_x M$ is totally real, i.e. $JD_x^\perp \subset T_x^\perp M$ for each $x \in M$.

For any vector field X tangent to M , we put

$$X = TX + FX, \quad (1.7)$$

where TX and FX belong to the distribution D and $D^\perp$ respectively.

2. SECTIONAL CURVATURE OF CR-SUBMANIFOLDS.

Let M be a CR -submanifold of a l.c.k. space form $\bar{M}(c)$. Then using Gauss equation (1.6), the curvature tensor of M is given by

$$\begin{aligned}R(X, Y, Z, W) = & \frac{c}{4} [g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(JTX, W)g(JTY, Z) - g(JTX, Z)g(JTY, W) \\ & - 2g(JTX, Y)g(JTZ, W)] + \frac{3}{4} [P(X, W)g(Y, Z) - P(X, Z)g(Y, W) + g(X, W)P(Y, Z) - g(X, Z)P(Y, W)] \\ & + \frac{1}{4} [P(X, JTW)g(JTY, Z) - P(X, JTZ)g(JTY, W) + g(JTX, W)P(Y, JTZ) - g(JTX, Z)P(Y, JTW) \\ & - 2g(JTZ, W)P(X, JTY) - 2P(Z, JTW)g(JTX, Y)] + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W))\end{aligned}\quad (2.1)$$

for $X, Y, Z, W \in TM$.

Let $\bar{H}(X)$ be the holomorphic sectional curvature of M determined by a unit vector X and JX . Then from (1.5) we have

$$\bar{H}(X) = \bar{R}(X, JX, JX, X) = -\frac{c}{2} + \frac{7}{4} P(X, X). \quad (2.2)$$

Now suppose that $\bar{K}(X \wedge Y)$ be the sectional curvature of $\bar{M}$ determined by a unit vector X and JX . Then from (1.5) we have

$$\bar{K}(X \wedge Y) = \bar{R}(X, Y, Y, X) = \frac{c}{4} [1 + g(JX, Y)^2 + 2g(JX, Y)] + \frac{3}{4} [P(X, X) + P(Y, Y)] + P(X, JY)g(JX, Y). \quad (2.3)$$

Next, suppose that $K(X \wedge Y)$ be the sectional curvature of M determined by orthonormal tangent vectors $\{X, Y\}$ of M . Then using (1.6) and (2.3), we have

$$\begin{aligned}K(X \wedge Y) = & \frac{c}{4} [1 + g(JTX, Y)^2 + 2g(JTX, Y)] + \frac{3}{4} [P(X, X) + P(Y, Y)] + P(X, JTY)g(JTX, Y) \\ & + g(h(X, X), h(Y, Y)) - \|h(X, Y)\|^2,\end{aligned}\quad (2.4)$$

for all X, Y tangent to M . From this, we have

PROPOSITION 2.1. Let M be a CR -submanifold of a l.c.k. space form $\bar{M}(c)$. If M is totally geodesic in $\bar{M}(c)$, then the sectional curvature of M is given by

$$K(X \wedge Y) = \frac{c}{4} [1 + g(JTX, Y)^2 + 2g(JTX, Y)] + \frac{3}{4} [P(X, X) + P(Y, Y)] + P(X, JTY)g(JTX, Y) \quad (2.5)$$

for all X, Y tangent to M .

DEFINITION 2.1. A CR -submanifold M of a l.c.k. space form $\bar{M}(c)$ is said to be D -totally (resp. $D^\perp$ -totally geodesic) if $h(X, Y) = 0$ (resp. $h(Z, W) = 0$) for all $X, Y \in D, (Z, W \in D^\perp)$.

Thus as an immediate consequence of (2.5) we have

COROLLARY 2.2. Let M be a CR -submanifold of a l.c.k. space form $\bar{M}(c)$. If M is $D^\perp$ -totally geodesic in $\bar{M}(c)$, then the sectional curvature of M is given by

$$K(X \wedge Y) = \frac{c}{4} + \frac{3}{4}[P(X, X) + P(Y, Y)] \quad \text{for all } X, Y \in D. \quad (2.6)$$

The holomorphic sectional curvature H of M determined by a unit vector $X \in D$ is the sectional curvature determined by $\{X, JX\}$. Hence from (2.2), we have

$$H(X) = -\frac{c}{2} + \frac{7}{4}P(X, X) + g(h(X, X), h(JX, JX)) - \|h(X, JX)\|^2. \quad (2.7)$$

LEMMA [1]. Let M be a CR -submanifold of a Kaehler manifold $\bar{M}$. Then the holomorphic distribution D is involutive if and only if

$$h(JX, Y) = h(X, JY), \quad \forall X, Y \in D. \quad (2.8)$$

Making use of (2.8) in (2.7), we have

PROPOSITION 2.3. Let M be a CR -submanifold of a l.c.k.-space form $\bar{M}(c)$ with involutive distribution D , then

$$H(X) \leq \frac{7}{4}P(X, X), \quad \forall X \in D.$$

Moreover from (2.7), we have

PROPOSITION 2.4. A CR -submanifold M of a l.c.k. space form $\bar{M}(c)$ is D -totally geodesic if and only if the following conditions are satisfied:

(a) the holomorphic distribution D is involutive, and (b) $H(X) = \frac{7}{4}P(X, X) - \frac{c}{2}, \quad \forall X \in D$.

Let $\{E_1, E_2, \dots, E_m\}$ be a local field of orthogonal frames of M such that $\{E_1, E_2, \dots, E_P, E_{P+1} = JE_1, \dots, E_{2P} = JE_P\}$ (resp. $\{E_{2P+1}, \dots, E_{2P+q}\}$) is a local field of frames in D (resp. $D^\perp$).

DEFINITION 2.2. A CR -submanifold M is called D -minimal (resp. $D^\perp$ -minimal) if

$$\sum_{i=1}^{2P} h(E_i, E_i) = 0, \quad (\text{resp. } \sum_{i=1}^q h(E_{2P+i}, E_{2P+i}) = 0).$$

Thus we have,

PROPOSITION 2.5. Let M be a $D^\perp$ -minimal CR -submanifold of a l.c.k. space form $\bar{M}(c)$. Then M is D -totally geodesic if and only if

$$K(X \wedge Y) = \frac{1}{4}[c + 3(P(X, X) + P(Y, Y))], \quad \forall X, Y \in D.$$

3. RICCI TENSOR AND SCALAR CURVATURE OF CR-SUBMANIFOLDS.

Let S be the Ricci tensor and ρ the scalar curvature of M . Then

$$S(X, Y) = \sum_i R(E_i, X; Y, E_i), \quad \rho = \sum_j S(E_j, E_j),$$

for any vector fields X, Y tangent to M . By the straight forward calculation from (2.1), we get

$$\begin{aligned} S(X, Y) &= \frac{c}{4}(m+2)g(X, Y) + \frac{3}{4} \sum_{i=1}^m \{P(E_i, E_i)g(X, Y) - P(E_i, Y)g(X, E_i) + mP(X, Y) - P(X, E_i)g(Y, E_i)\} \\ &\quad - \frac{5}{4} \sum_{i=1}^m \{P(JY, E_i)g(JX, E_i) + P(JX, E_i)g(JY, E_i)\} + \sum_{i=1}^m \{g(h(X, Y), h(E_i, E_i)) - g(h(E_i, X), h(E_i, Y))\}, \end{aligned}$$

since $g(JTE_i, E_i) = 0$.

The scalar curvature is given by

$$\rho = \frac{c}{4}m(m+2) + \sum_{i,j=1}^m \{g(h(E_j, E_j), h(E_i, E_i)) - g(h(E_i, E_j), h(E_i, E_j))\}.$$

Thus we have

PROPOSITION 3.1. Let M be a minimal CR -submanifold of a l.c.k. space form, then we have

$$(a) \quad S(X, Y) - \frac{c}{4}(m+2)g(X, Y) - \frac{3}{4} \sum_{i=1}^m \{P(E_i, E_i)g(X, Y) - P(E_i, Y)g(X, E_i) + mP(X, Y) - P(X, E_i)g(Y, E_i)\} \\ + \frac{5}{4} \sum_{i=1}^m \{P(JY, E_i)g(JX, E_i) + P(JX, E_i)g(JY, E_i)\}$$

is semi-definite for all $X, Y \in D$.

$$(b) \quad \rho \leq \frac{c}{4}m(m+2).$$

Similarly we have:

PROPOSITION 3.2. Let M be a minimal CR -submanifold of a l.c.k.-space form. Then M is totally geodesic if and only if

$$(a) \quad S(X, Y) = (m+2)g(X, Y) + \frac{3}{4} \sum_{i=1}^m \{P(E_i, E_i)g(X, Y) - P(E_i, Y)g(X, E_i) + mP(X, Y) \\ - P(X, E_i)g(Y, E_i)\} - \frac{5}{4} \sum_{i=1}^m \{P(JY, E_i)g(JX, E_i) + P(JX, E_i)g(JY, E_i)\} \\ (b) \quad \rho = m(m+2).$$

ACKNOWLEDGEMENT. The author expresses his sincere thanks to the referee for his suggestions.

REFERENCES

1. BEJANCU, A., CR -submanifolds of a Kaehler manifold I, Amer. Math. Soc. **69** (1978), 135-142.
2. BEJANCU, A., CR -submanifolds of a Kaehler manifold II, Trans. Amer. Soc. **250** (1979), 333-345.
3. MATSUMOTO, K., On CR -submanifolds of locally conformal Kaehler manifold, J. Korean Math. Soc. **21** (1984).
4. VAISMAN, I., On locally and globally conformal Kaehler manifolds, Trans. Amer. Math. Soc. **262** (1980), 533-542.
5. CHEN, B.Y., CR -submanifolds of a Kaehlerian manifold I, J. Diff. Geom. **16** (1981), 305-322.
6. YANO, K. & KON, M., Differential geometry of CR -submanifolds, Geometrical Dedicata **10** (1981), 369-391.