A CRITERION FOR P-VALENTLY STARLIKE FUNCTIONS

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(Received December 22, 1992 and in revised form April 19, 1993)

ABSTRACT. The object of the present paper is to prove a criterion for p-valently starlike functions in the open unit disk.

KEY WORDS AND PHRASES. Analytic, open unit disk, p-valently starlike.

1991 AMS SUBJECT CLASSIFICATION CODE. Primary, 30C45.

1. INTRODUCTION.

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} \setminus \{1, 2, 3, \ldots\}),$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function $f(z)$ belonging to $A(p)$ is said to be p-valently starlike in $U$ if it satisfies

$$Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in U).$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions $f(z)$ which are p-valently starlike in $U$ (cf. [1]).

Recently, Nunokawa [4] has shown that

**THEOREM A.** If $f(z) \in A(p)$ satisfies $f(z) \not= 0 (0 < |z| < 1)$ and

$$Re \left\{ \frac{1 + zf''(z)}{zf'(z)} \right\} < 1 + \frac{1}{z^p} \quad (z \in U),$$

then $f(z) \in S(p)$.

In the present paper, we derive a new criterion for the class $S(p)$ involving the above result by Nunokawa [4].

2. A NEW CRITERION.

To derive our main result, we have to recall here the following lemma due to Jack [2] (also, due to Miller and Mocanu [3]).

**LEMMA.** Let $w(z)$ be analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0$, then we can write

$$z_0 w'(z_0) = kw(z_0),$$

where $k$ is a real number and $k \geq 1$. 
Now, we prove

THEOREM. If \( f(z) \in A(p) \) satisfies \( f(z) \neq 0(0 < |z| < 1) \) and

\[
\left| \arg \left\{ \frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \left( 1 + \frac{1}{4p} \right) \right\} \right| > 0 \quad (z \in U),
\]

then \( f(z) \in S(p) \) and

\[
\left| \frac{zf'(z)}{f(z)} - p \right| < p \quad (z \in U).
\]

PROOF. Define the function \( w(z) \) by

\[
\frac{zf'(z)}{f(z)} = p(1 + w(z)).
\]

Then \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). It follows from (2.4) that

\[
1 + \frac{zf''(z)}{f'(z)} = p(1 + w(z)) + \frac{zw'(z)}{1 + w(z)},
\]

so that,

\[
\frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) = 1 + \frac{zw'(z)}{p(1 + w(z))^2}.
\]

Suppose that there exists a point \( z_0 \in U \) such that

\[
0 < \max \left| z \right| \leq \left| z_0 \right| \left| w(z) \right| = \left| w(z_0) \right| = 1 \quad (w(z_0) \neq -1).
\]

Then, applying Lemma, we can write

\[
z_0 w'(z_0) = kw(z_0) \quad (k \geq 1)
\]

and \( w(z_0) = e^{i\theta} (\theta \neq \pi) \). Thus we have

\[
\frac{f(z_0)}{z_0 f'(z_0)} \left( 1 + \frac{zf''(z_0)}{f'(z_0)} \right) = 1 + \frac{k e^{i\theta}}{p(1 + e^{i\theta})^2}
\]

\[
= 1 + \frac{k}{2p(1 + \cos \theta)} \geq 1 + \frac{1}{4p}.
\]

Note that the condition (2.2) implies

\[
\frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \neq \alpha \quad (z \in U),
\]

where \( \alpha \geq 1 + 1/4p \). Therefore, (2.7) contradicts our condition (2.2). Consequently, we conclude that

\[
\left| \frac{zf'(z)}{f(z)} - p \right| < p \quad (z \in U),
\]

that is, that \( f(z) \in S(p) \).

Letting \( p = 1 \) in Theorem, we have

COROLLARY. If \( f(z) \in A(1) \) satisfies \( f(z) \neq 0(0 < |z| < 1) \) and

\[
\left| \arg \left\{ \frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \frac{5}{4} \right\} \right| > 0 \quad (z \in U),
\]

then \( f(z) \in S(1) \) and

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in U).
\]
ACKNOWLEDGEMENT. The research of the first author was supported in part by Japanese Ministry of Education, Science and Culture under Grant-in-Aid for General Scientific Research (No. 04640204).

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