MAXIMUM PRINCIPLES FOR PARABOLIC SYSTEMS COUPLED IN BOTH FIRST-ORDER AND ZERO-ORDER TERMS

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ABSTRACT. Some generalized maximum principles are established for linear second-order parabolic systems in which both first-order and zero-order terms are coupled.

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1. INTRODUCTION.

Hile and Protter [2] proved that the Euclidean length of the solution vector $u \in C^2(D) \cap C(\bar{D})$ of the second-order elliptic system

$$\sum_{i,k=1}^n a_{ik}(x) \frac{\partial^2 u_s}{\partial x_i \partial x_k} + \sum_{i=1}^n \sum_{j=1}^m b_{si,j}(x) \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^m c_{sj}(x) u_j = 0, \quad s = 1, \dots, m,$$

can be bounded by a constant times the maximum of its boundary values under a "small" condition which requires that either the domain D or the coefficients $b_{si,j}$ and c_{sj} are sufficiently small. In this paper, we have established the same kind of maximum principle for the second-order parabolic system

$$\sum_{i,k=1}^n a_{ik}(x,t) \frac{\partial^2 u_s}{\partial x_i \partial x_k} - \frac{\partial u_s}{\partial t} + \sum_{i=1}^n \sum_{j=1}^m b_{si,j}(x,t) \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^m c_{sj}(x,t) u_j = 0, 1 \leq s \leq m.$$

Moreover, our parabolic version of the maximum principle holds without any "small" conditions.

When the coupling occurs only in the zero-order terms (i.e., in the case of $b_{si,j} = 0$ for all i, j, s except when $j = s$), the above systems are called weakly coupled systems. For weakly coupled second-order parabolic systems, similar maximum principles have been obtained by Stys [4] and Zhou [6]. Under different assumptions, different maximum principles in which the components rather than the Euclidean length of the solution vector are bounded can be found in Protter and Weinberger [3] and Dow [1]. In Weinberger's paper [5], both kinds of maximum principles have been reformulated and studied in terms of invariant sets.

2. MAIN RESULTS.

Consider a second-order parabolic operator with real coefficients,

$$M \equiv \sum_{i,k=1}^n a_{ik}(x,t) \frac{\partial^2}{\partial x_i \partial x_k} - \frac{\partial}{\partial t}, \quad a_{ij} = a_{ji}$$

in a general bounded domain Ω in $\mathbb{R}^n \times \mathbb{R}_t$ ($n \geq 1$) with the boundary $\partial\Omega = \partial_p\Omega \cup \partial_t\Omega$. Here $\partial_p\Omega$ is the parabolic boundary of Ω and $\partial_t\Omega = \partial\Omega \setminus \partial_p\Omega$. We suppose that $\Omega \subset D \times (0, T)$ where D is a

bounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. The operator M is assumed to be uniformly parabolic in Ω ; i.e., there is a constant $\delta > 0$ such that for all $(x, t) \in \Omega$ and all $(y_1, \dots, y_n)$ in $\mathbb{C}^n$ the inequality

$$\sum_{i,k=1}^n a_{ik}(x, t) y_i \bar{y}_k \geq \delta \sum_{i=1}^n |y_i|^2 \quad (2.1)$$

holds. The operator M is the principal part of each equation in the second-order parabolic system

$$Mu_s + \sum_{i=1}^n \sum_{j=1}^m b_{sij}(x, t) \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^m c_{sj}(x, t) u_j = 0, \quad s = 1, 2, \dots, m. \quad (2.2)$$

We suppose that the complex-valued coefficients b_{sij} , c_{sj} have the property that for all $\xi \in \mathbb{C}^m$ and all $(x, t) \in \Omega$.

$$\sum_{r,s=1}^m \left[c_{sr} + \bar{c}_{rs} + \frac{1}{2} \sum_{j=1}^m \sum_{k,i=1}^n A_{ki} b_{sij} \bar{b}_{rkj} \right] \xi_r \bar{\xi}_s \leq K |\xi|^2, \text{ for some } K > 0. \quad (2.3)$$

Here $(A_{ki}) = (A_{ik})$ denotes the inverse matrix of (a_{ik}) . A solution $u = (u_1, u_2, \dots, u_m)$ is a complex-valued $C^{2,1}(\Omega \cup \partial_t \Omega) \cap C(\bar{\Omega})$ function which satisfies (2) in Ω . Here $C^{k,h}(\Omega)$ is defined as the set of functions $f(x, t)$ having all x (space) derivatives of order $\leq k$ and t (time) derivatives of order $\leq h$ continuous in Ω .

THEOREM 1. Assume conditions (1.1) and (1.3) hold. If u is a solution of (2.2) and α is a positive $C^{2,1}(\Omega \cup \partial_t \Omega)$ function, then the product $\alpha |u|^2 = \alpha \sum_{j=1}^m |u_j|^2$ cannot attain a positive maximum at any point in $\Omega \cup \partial_t \Omega$ where α satisfies

$$\alpha^{-1} M\alpha - 2\alpha^{-2} \sum_{i,k=1}^n a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} > K. \quad (2.4)$$

PROOF. We set $p = |u|^2 = \sum_{s=1}^m |u_s|^2$ and find

$$M(\alpha p) = pM\alpha + \alpha Mp + 2 \sum_{i,k=1}^n a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial p}{\partial x_k}. \quad (2.5)$$

At a point $(x, t) \in \Omega \cup \partial_t \Omega$ where αp attains a maximum, we have

$$0 \leq \frac{\partial(\alpha p)}{\partial t}, \quad 0 = \frac{\partial(\alpha p)}{\partial x_k} = \alpha \frac{\partial p}{\partial x_k} + p \frac{\partial \alpha}{\partial x_k}, \quad 1 \leq k \leq n,$$

and (2.5) becomes

$$M(\alpha p) = p \left[M\alpha - 2\alpha^{-1} \sum_{i,k=1}^n a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} \right] + \alpha Mp. \quad (2.6)$$

A direct computation yields

$$\begin{aligned} Mp &= \sum_{s=1}^m \left[u_s M\bar{u}_s + \bar{u}_s Mu_s + 2 \sum_{i,k=1}^n a_{ik} \frac{\partial u_s}{\partial x_i} \frac{\partial \bar{u}_s}{\partial x_k} \right] \\ &= \sum_{s=1}^m \left\{ -2\operatorname{Re} \left[\bar{u}_s \left(\sum_{j=1}^n \sum_{j=1}^m b_{sij} \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^m c_{sj} u_j \right) \right] + 2 \sum_{i,k=1}^n a_{ik} \frac{\partial u_s}{\partial x_i} \frac{\partial \bar{u}_s}{\partial x_k} \right\} \\ &= 2 \left\{ \sum_{j=1}^m \sum_{i,k=1}^n a_{ik} \left[\frac{\partial u_j}{\partial x_i} - \frac{1}{2} \sum_{q=1}^n \sum_{r=1}^m A_{iq} \bar{b}_{rjq} u_r \right] \left[\frac{\partial \bar{u}_j}{\partial x_k} - \frac{1}{2} \sum_{q=1}^n \sum_{s=1}^m A_{kq} b_{sqj} \bar{u}_s \right] \right. \\ &\quad \left. - \frac{1}{4} \sum_{r,s=1}^m \left[\sum_{k,q=1}^n \sum_{j=1}^m A_{kq} b_{sqj} \bar{b}_{rkj} \right] u_r \bar{u}_s \right\} - \sum_{r,s=1}^m (c_{sr} + \bar{c}_{rs}) u_r \bar{u}_s \\ &\geq -K \sum_{s=1}^m |u_s|^2 = -Kp. \end{aligned}$$

Hence, from (2.6), we have

$$M(\alpha p) \geq \alpha p \left[\alpha^{-1} M \alpha - 2 \alpha^{-2} \sum_{i,k=1}^n a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} - K \right]. \quad (2.7)$$

This inequality holds at any point in $\Omega \cup \partial_t \Omega$ where αp attains a maximum. Thus αp cannot achieve a positive maximum at any point in $\Omega \cup \partial_t \Omega$ where the quantity in brackets in (2.7) is positive. The theorem is established. $\square$

REMARK. If for all $(x, t) \in \Omega$,

$$|c_{sj}| \leq K_0, |b_{sj}| \leq K_1, 1 \leq i \leq n, 1 \leq j, s \leq m, \text{ for some } K_0, K_1 \in \mathbb{R}, \quad (2.8)$$

then for any $\xi \in \mathbb{C}^m$,

$$\begin{aligned} & \sum_{r,s=1}^m \left[c_{sr} + \bar{c}_{rs} + \frac{1}{2} \sum_{j=1}^m \sum_{k,i=1}^n A_{ki} b_{sj} \bar{b}_{rk} \right] \xi_r \bar{\xi}_s \\ & \leq \sum_{r,s=1}^m |c_{sr}| (|\xi_r|^2 + |\xi_s|^2) + \frac{1}{2} \sum_{j=1}^m \sum_{k,i=1}^n A_{ki} \left(\sum_{s=1}^m b_{sj} \bar{\xi}_s \right) \left(\sum_{r=1}^m \bar{b}_{rk} \xi_r \right) \\ & \leq 2mK_0 \sum_{s=1}^m |\xi_s|^2 + \frac{1}{2\delta} \sum_{j=1}^m \sum_{i=1}^n \left| \sum_{s=1}^m b_{sj} \bar{\xi}_s \right|^2 \\ & \leq 2mK_0 |\xi|^2 + \frac{m}{2\delta} \sum_{j=1}^m \sum_{i=1}^n \sum_{s=1}^m |b_{sj} \bar{\xi}_s|^2 \leq [2mK_0 + (2\delta)^{-1} nm^2 K_1^2] |\xi|^2, \end{aligned}$$

which is the condition (2.3) with $K := 2mK_0 + (2\delta)^{-1} nm^2 K_1^2$. Hence, the single bound (2.3) in Theorem 1 can be replaced by the separate bounds (2.8) with $K := 2mK_0 + (2\delta)^{-1} nm^2 K_1^2$.

Under the conditions (2.1) and (2.3) (or (2.1) and (2.8)), by choosing $\alpha(x, t) = e^{-(K+\varepsilon)t}$, $\varepsilon > 0$, the condition (2.4) will be satisfied. Hence from Theorem 1, we get the following maximum principle:

COROLLARY 2 (Maximum Principle). For any solution u of the system (2.2), the function

$$|u(x, t)|^2 \exp[-(K + \varepsilon)t], \varepsilon > 0,$$

does not attain a positive maximum in $\Omega \cup \partial_t \Omega$, and

$$\|u\|_{0,\Omega} \leq \exp(KT/2) \|u\|_{0,\partial_p \Omega}. \quad (2.9)$$

Here $K = (2\delta)^{-1} nm^2 K_1^2 + 2mK_0$ and $\|u\|_{0,\Omega} := \sup_{(x,t) \in \Omega} |u(x, t)|$.

REMARK. Results similar to Theorem 1 and Corollary 2 for second-order elliptic systems were proven by Hile and Protter [2] (under a condition which is similar to (2.8)). But their maximum principle for elliptic systems only holds under the restriction that either the domain D is sufficiently small or the coefficients of the elliptic system are restricted sufficiently. Corollary 2 tells us that these restrictions can be lifted for parabolic systems.

COROLLARY 3 (Uniqueness). The system (2.2) with the initial-boundary condition

$$u|_{\partial_p \Omega} = \varphi(x, t)$$

has at most one solution $u \in C^{2,1}(\Omega \cup_t \Omega) \cap C(\bar{\Omega})$.

Theorem 1 can be used to obtain bounds on the gradient of the $C^{3,2}$ solution of the parabolic system (2.2), provided the coefficients are C^1 and

$$\|a_{ik}\|_{1,\Omega} \leq L_2, \|b_{sj}\|_{1,\Omega} \leq L_1, \|c_{sj}\|_{1,\Omega} \leq L_0, \text{ for some } L_2, L_1, L_0 \in \mathbb{R}. \quad (2.10)$$

Here $\|f\|_{1,\Omega} := \|f\|_{0,\Omega} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{0,\Omega} + \left\| \frac{\partial f}{\partial t} \right\|_{0,\Omega}$.

We differentiate (2.2) with respect to x_k and t , and get $m(n+1)$ equations:

$$\begin{aligned}
M\left(\frac{\partial u_s}{\partial x_h}\right) &+ \sum_{i,k=1}^n \frac{\partial a_{ik}}{\partial x_h} \frac{\partial}{\partial x_i} \left(\frac{\partial u_s}{\partial x_k}\right) + \sum_{i=1}^n \sum_{j=1}^m b_{sj} \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_h}\right) \\
&+ \sum_{i=1}^n \sum_{j=1}^m \frac{\partial b_{sj}}{\partial x_h} \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^m c_{sj} \frac{\partial u_j}{\partial x_h} + \sum_{j=1}^m \frac{\partial c_{sj}}{\partial x_h} u_j = 0, \\
&s = 1, 2, \dots, m \text{ and } h = 1, 2, \dots, n;
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
M\left(\frac{\partial u_s}{\partial t}\right) &+ \sum_{i,k=1}^n \frac{\partial a_{ik}}{\partial t} \frac{\partial}{\partial x_i} \left(\frac{\partial u_s}{\partial x_k}\right) + \sum_{i=1}^n \sum_{j=1}^m b_{sj} \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial t}\right) \\
&+ \sum_{i=1}^n \sum_{j=1}^m \frac{\partial b_{sj}}{\partial t} \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^m c_{sj} \frac{\partial u_j}{\partial t} + \sum_{j=1}^m \frac{\partial c_{sj}}{\partial t} u_j = 0, \\
&h = 1, 2, \dots, n.
\end{aligned} \tag{2.12}$$

By combining (2.2), (2.11) and (2.12) we get a system (of the form (2.2)) consisting of $m(n+2)$ equations in the $m(n+2)$ unknowns $u_s, \frac{\partial u_s}{\partial x_h}, \frac{\partial u_s}{\partial t}, s = 1, 2, \dots, m, h = 1, 2, \dots, n$.

THEOREM 4. Let $K := (2\delta)^{-1}n(n+2)^2m^2(\text{Max}\{L_1, L_2\})^2 + 2m(n+2)\text{Max}\{L_0, L_1\}$ and suppose that u is a $C^{3,2}(\Omega \cup \partial_t \Omega) \cap C^1(\bar{\Omega})$ solution of (2.2) and α is a positive $C^{2,1}(\Omega \cup \partial_t \Omega)$ function. Then the product

$$\alpha(x, t)[|u(x, t)|^2 + |\nabla u(x, t)|^2] = \alpha(x, t) \sum_{s=1}^m \left[|u_s|^2 + \sum_{i=1}^n \left| \frac{\partial u_s}{\partial x_i} \right|^2 + \left| \frac{\partial u_s}{\partial t} \right|^2 \right]$$

cannot attain a positive maximum at any point in $\Omega \cup \partial_t \Omega$ where α satisfies (2.4).

COROLLARY 5. Let K be the same number of Theorem 4. Then, for any $C^{3,2}(\Omega \cup \partial_t \Omega) \cap C^1(\bar{\Omega})$ solution u of the system (2.2), we have

$$\|u\|_{0,\Omega}^2 + \|\nabla u\|_{0,\Omega}^2 \leq \exp(KT) (\|u\|_{0,\partial_p \Omega}^2 + \|\nabla u\|_{0,\partial_p \Omega}^2),$$

or equivalently,

$$\|u\|_{1,\Omega} \leq \exp(KT/2) \cdot \|u\|_{1,\partial_p \Omega}.$$

REMARK. Under the condition that either $(c_{sj})_{m \times m}$ is a constant matrix or $(c_{sj})_{m \times m}$ is invertible for all $(x, t) \in \Omega$, the unknowns $u_s, s = 1, \dots, m$, can be eliminated from the system (2.2), (2.11), (2.12), and then a system of $m(n+1)$ equations in the gradient of u yields a maximum principle for $\alpha |\nabla u|^2$.

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