VECTOR-VALUED MEANS AND WEAKLY ALMOST PERIODIC FUNCTIONS

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ABSTRACT. A formula is set up between a vector-valued mean and scalar-valued means that enables us translate many important results about scalar-valued means developed in [1] to vector-valued means. As applications of the theory of vector-valued means, we show that the definitions of a mean in [2] and [3] are equivalent and the space of vector-valued weakly almost periodic functions is admissible.

KEY WORDS AND PHRASES. Means, semigroup, weakly almost periodic functions

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Scalar-valued means have been much studied. However, little has been done on the vector-valued means. In this paper we develop the theory of vector-valued means.

In Lemma 1.4, we set up a formula between a vector-valued mean and scalar-valued means, by which we will be able to translate many important results about scalar-valued means developed in [1] to vector-valued means. We present these results in Sections 1, 2 and 3. As an application of the theory established in these sections, we investigate vector-valued weakly almost periodic functions in Section 4.

§1. Means on a Linear Subspace of \( B(S, X) \)

Throughout this paper, \( S \) denotes a semigroup which need not have an identity, \( X \) denotes a Banach space and \( X^* \) is the dual space of \( X \). \( B(S, X) \) denotes all of the bounded functions from \( S \) to \( X \). When \( X = \mathbb{C} \), we simply write \( B(S) \) for \( B(S, X) \). \( A \) denotes a linear subspace of \( B(S, X) \) containing the constant functions. \( L(A, X) \) denotes all of the bounded linear mappings from \( A \) to \( X \).

Let \( f \in B(S, X) \). Then the right (respectively, left) translate \( R_s f \) of \( f \) by \( s \in S \) is the map \( R_s f(t) = f(ts) \) (respectively, \( L_s f(t) = f(st) \)) for all \( t \in S \).

\( A \) is said to be right (respectively, left) translation invariant if \( R_s A = \{ R_s f : s \in S, f \in A \} \subset A \) (respectively, \( L_s A = \{ L_s f : s \in S, f \in A \} \subset A \)). \( A \) is said to be translation invariant if it is both right and left translation invariant.

Definition 1.1 [2]. A linear mapping \( \mu : A \to X \) is called a mean on \( A \) provided \( \mu(f) \in \overline{\text{co}} f(S) \), for all \( f \in A \). Denote by \( M(A) \) the set of all means on \( A \).

If \( A \) is right (respectively, left) translation invariant, \( \mu \) is said to be right (respectively, left) invariant if \( \mu(R_s f) = \mu(f) \) (respectively, \( \mu(L_s f) = \mu(f) \)) for all \( s \in S \) and \( f \in A \).
Remark 1.2. It follows from [1, 2.1.2] that Definition 1.1 will reduce to the definition of a scalar-valued mean when $X = \mathbb{C}$.

Of course, the evaluation mapping $\epsilon : S \to \mathcal{L}(A, X)$, defined by

$$\epsilon(s)(f) = f(s) \quad (s \in S, \ f \in A)$$

is in $M(A)$, and if $\mu \in M(A)$ and $f \in A$ is a constant function, then $\mu(f)$ is the constant.

The following proposition is obvious.

Proposition 1.3. If $A$ is a linear subspace of $\mathcal{B}(S, X)$ containing the constant functions, then each $\mu \in M(A)$ is in $\mathcal{L}(A, X)$ with $\|\mu\| = 1$.

For each $x^* \in X^*$,

$$x^*A = \{x^*f = x^* \circ f : f \in A\}$$

is a linear subspace of $\mathcal{B}(S)$.

Here we have adopted the definition in [2] of a mean on $A$. [3] gives a definition of a mean in terms of a scalar-valued mean on $\overline{\mathcal{P}}(X^* \circ A) = \overline{\mathcal{P}}\{x^*A : x^* \in X^*\}$. In the next lemma, we set up a connection like this, and we will show in Theorem 1.7 that the definitions of a mean in [2] and [3] are equivalent. We will deal with other applications in §4.

Lemma 1.4. Let $A$ be a linear subspace of $\mathcal{B}(S, X)$. A mapping $\mu : A \to X$ is in $M(A)$ if and only if, for each $x^* \in X^*$, there is a $\varphi_{\mu, x^*} \in M(x^*A)$ such that

$$x^*\mu(f) = \varphi_{\mu, x^*}(x^*f) \quad (f \in A).$$

If $A$ is right (left) translation invariant, then $\mu$ is right (left) invariant if and only if the $\varphi_{\mu, x^*}$'s are right (left) invariant. Furthermore, the set $\varphi_\mu = \{\varphi_{\mu, x^*} : x^* \in X^*\}$ is uniquely determined by $\mu$, i.e., $\varphi_{\mu, x^*} = \varphi_{\mu', x^*}$ for all $x^* \in X^*$ if and only if $\mu = \mu'$.

Proof. Sufficiency. First, $\mu$ is a linear mapping from $A$ to $X$. In fact, for $f, g \in A$ and $\alpha, \beta \in \mathbb{C}$,

$$x^*\mu(\alpha f + \beta g) = \varphi_{\mu, x^*}(x^*(\alpha f + \beta g))$$

$$= \varphi_{\mu, x^*}(x^*(\alpha f)) + \varphi_{\mu, x^*}(x^*(\beta g))$$

$$= \alpha \varphi_{\mu, x^*}(x^*f) + \beta \varphi_{\mu, x^*}(x^*g)$$

$$= \alpha x^*\mu(f) + \beta x^*\mu(g)$$

$$= x^*(\alpha \mu(f) + \beta \mu(g)).$$

The equality is true for all $x^* \in X^*$, therefore

$$\mu(\alpha f + \beta g) = \alpha \mu(f) + \beta \mu(g).$$

We claim that $\mu(f) \in \overline{\mathcal{P}}f(S)$, for all $f \in A$. If it is not true for some $f \in A$, by the Hahn–Banach theorem there is an $x^* \in X^*$ such that
\[ |x^*\mu(f)| > \sup_{s \in S}|x^*f(s)| = \|x^*f\|. \]

It follows from Remark 1.2 and Proposition 1.3 that \( \varphi_{\mu,x^*} \in M(x^*A) \) is in \((x^*A)^*\) with \( \|\varphi_{\mu,x^*}\| = 1 \). So
\[ |x^*\mu(f)| = |\varphi_{\mu,x^*}(x^*f)| \leq \|x^*f\|, \]
a contradiction.

**Necessity.** For each \( x^* \in X^* \), define \( \varphi_{\mu,x^*} \in (x^*A)^* \) by
\[ \varphi_{\mu,x^*}(x^*f) = x^*\mu(f) \quad (f \in A). \]
\( \varphi_{\mu,x^*} \) is well-defined on \( x^*A \). For, if \( x^*f = 0 \) for some \( f \in A \), then \( f(S) \subset N(x^*) \), the null subspace of \( x^* \), so \( \varphi_{\mu,x^*}(x^*f) = x^*\mu(f) = 0 \) since \( \mu(f) \in \overline{\partial f}(S) \) (Definition 1.1). Clearly \( \varphi_{\mu,x^*} \) is linear on \( x^*A \). Furthermore
\[ \varphi_{\mu,x^*}(x^*f) = x^*(\overline{\partial f}(S)) \subset \overline{\partial x^*f}(S), \]
so \( \varphi_{\mu,x^*} \) is in \( M(x^*A) \).

The rest of the lemma is clear.

We can furnish \( L(A,X) \) with two topologies, both of which make \( L(A,X) \) a locally convex topological space. One is the strong operator topology \( \tau_s \), which is the weakest topology of \( (A,X) \) relative to which the mapping \( U \to UF : L(A,X) \to X \) is continuous for each \( f \in A \), and the other is the weak operator topology \( \tau_w \), which is the weakest topology of \( L(A,X) \) relative to which the mapping \( U \to U^*f : L(A,X) \to C \) is continuous for each \( f \in A \) and \( x^* \in X^* \). These topologies can be relativized to \( M(A) \subset L(A,X) \).

**Proposition 1.5.** Let \( A \) be a linear subspace of \( B(S,X) \). Then, for \( \tau_s \)

1. \( M(A) \) is convex and closed in \( L(A,X) \);
2. \( co(\epsilon(S)) \) is dense in \( M(A) \);
3. if \( S \) is a topological space and \( A \subset C(S,X) \), then \( \epsilon : S \to M(A) \) is continuous.

Furthermore, if the range \( f(S) \) of \( f \) is relatively compact in \( X \) for each \( f \in A \), then \( M(A) \) is \( \tau_s \)-compact.

**Proof.**

1. The convexity of \( M(A) \) follows directly from Definition 1.1. To show that \( M(A) \) is closed, let \( \{\mu_\alpha\} \subset M(A) \) converge to \( \mu \in L(A,X) \) for \( \tau_s \). Then \( \mu_\alpha(f) \to \mu(f) \) for each \( f \in A \), and since \( \mu_\alpha(f) \in \overline{\partial f}(S) \) for all \( \alpha \), \( \mu(f) \in \overline{\partial f}(S) \). Therefore, \( \mu \in M(A) \).
2. Clearly, \( co(\epsilon(S)) \subset M(A) \). If there is a \( \mu \in M(A) \) such that \( \mu \notin \overline{\partial}(\epsilon(S)) \), the closure being taken in \( \tau_s \), then there is an \( f \in A \) such that \( \mu(f) \notin \overline{\partial}(\epsilon(S)f) = \overline{\partial f}(S) \), which contradicts Definition 1.1.
3. is obvious.
The proof of the compactness of $M(A)$, if $A$ satisfies the compactness condition, is similar to that of its counterpart in the following proposition, so we omit it.

**Proposition 1.6.** Let $A$ be a linear subspace of $\mathcal{B}(S, X)$. Then the conclusions (1)-(3) of the previous proposition are true for $\tau_w$. Furthermore, if $A$ is such that the range $f(S)$ of $f$ is weakly relatively compact in $X$ for each $f \in A$, then $M(A)$ is $\tau_w$-compact.

**Proof.** Using Lemma 1.4, we can prove (1)-(3) in much the same way that (1)-(3) of Proposition 1.5 we proved.

We now show that $M(A)$ is $\tau_w$-compact when $A$ satisfies the weak compactness condition. For each $x^* \in X^*$, $M(x^*A)$ is weak$^*$ compact [1, 2.1.8]. Therefore, the product space

$$\prod := \prod \{M(x^*A) : x^* \in X^*\}$$

is compact in the product topology.

By Lemma 1.4, the mapping $\mu \rightarrow \varphi_\mu = \{\varphi_{\mu, x^*} : x^* \in X^*\} : M(A) \rightarrow \prod$ is 1-1, and it is homeomorphism when $M(A)$ has the topology $\tau_w$. To show that $M(A)$ is $\tau_w$-compact, it suffices to show that the image of $M(A)$ in $\prod$ is closed.

Let $\varphi = \{\varphi_{x^*} : x^* \in X^*\} \in \prod$ and let the image $\{\varphi_{\mu, x^*}\}$ of $\{\mu\}$ converge to $\varphi$ in $\prod$. We show that there is a $\mu \in M(A)$ such that $\varphi$ is the image of $\mu$ and $\mu_\alpha \rightarrow \mu$ in $\tau_w$.

Since $f(S)$ is weakly relatively compact in $X$ for each $f \in A$, by the Krein–Smulian theorem [1, A.10] $\overline{\partial f}(S)$ is weakly compact in $X$ for each $f \in A$. Since $\mu_\alpha(f) \in \overline{\partial f}(S)$ for all $\alpha$ and $x^*\mu_\alpha(f) \rightarrow \varphi_{x^*}(x^*f)$ for all $x^* \in X^*$, there is a $\mu(f) \in \overline{\partial f}(S)$ such that $x^*\mu(f) = \varphi_{x^*}(x^*f)$ for all $x^* \in X^*$. The map $f \rightarrow \mu(f)$ is clearly linear, so $\mu \in M(A)$. Thus $\mu_\alpha \rightarrow \mu$ in $\tau_w$, and the proof is complete.

The following theorem shows that the definition of a mean in [2] is equivalent to that in [3].

**Theorem 1.7.** A mapping $\mu : A \rightarrow X$ is in $M(A)$ if and only if there is a unique $\varphi_{\mu} \in M(\overline{\partial}(X^* \circ A))$ such that

$$x^*\mu(f) = \varphi_{\mu}(x^*f) \quad (x^* \in X^*, \ f \in A). \quad (1.1)$$

**Proof.** The sufficiency comes from the sufficiency in the first statement of Lemma 1.4.

Necessity. By Lemma 1.4, if $\mu$ is in $M(A)$, then for each $x^* \in X^*$ there is a $\varphi_{\mu, x^*}$ in $M(x^*A)$ such that

$$x^*\mu(f) = \varphi_{\mu, x^*}(x^*f) \quad (f \in A).$$

We show first that $\varphi_{\mu, x^*}$ is independent of $x^* \in X^*$, i.e., if $x_1^*, x_2^* \in X^*$ and $f_1, f_2 \in A$ are such that $x_1^*f_1 = x_2^*f_2$, then $\varphi_{\mu, x_1^*}(x_1^*f_1) = \varphi_{\mu, x_2^*}(x_2^*f_2)$.

Since $\mu \in M(A)$, by Proposition 1.6 (2) there is a net $\{\sum s \in S \lambda_\alpha(s) e(s)\}$ converging to $\mu$ for $\tau_w$; here each $\lambda_\alpha : S \rightarrow [0, 1]$ has finite support and satisfies $\sum s \in S \lambda_\alpha(s) = 1$. Next, $x_1^*(\sum s \in S \lambda_\alpha(s)f_1(s)) = x_2^*(\sum s \in S \lambda_\alpha(s)f_2(s))$ because $x_1^*f_1 = x_2^*f_2$, so
Therefore we can define \( \varphi \) for \( \sum_{i=1}^{m} \alpha_i x_i^* f_i \in \text{sp}(X^* \circ A) \) by

\[
\varphi(\sum_{i=1}^{m} \alpha_i x_i^* f_i) = \sum_{i=1}^{m} \alpha_i \varphi_{\mu, x_i^*}(x_i^* f_i).
\]

It is easy to see that \( \varphi \) is in \( M(\text{sp}(X^* \circ A)) \). Therefore \( \varphi \) has a unique extension to \( \overline{\text{sp}}(X^* \circ A) \) and satisfies (1.1).

The uniqueness is clear. The proof is finished.

By Theorem 1.7, we can write \( \varphi \) for \( \varphi_{\mu, x} \) in Lemma 1.4.

\section{Introversion and Semigroups of Vector-Valued Means}

**Definition 2.1.** Let \( A \) be a translation invariant linear subspace of \( B(S, X) \). For a linear map \( \mu \) from \( A \) to \( X \), define the left introversion operator \( T \mu : A \to B(S, X) \) by

\[
T \mu f(s) = \mu(L_s f) \quad (f \in A, \ s \in S)
\]

and analogously define the right introversion operator \( U \mu : A \to B(S, X) \) by

\[
U \mu f(s) = \mu(R_s f) \quad (f \in A, \ s \in S).
\]

If \( T \mu A \subset A \) for all \( \mu \in M(A) \), we will say that \( A \) is left introverted; we will say that \( A \) is right introverted if \( U \mu A \subset A \). \( A \) is introverted if it is both left and right introverted.

A semitopological semigroup \( S \) is a semigroup and a Hausdorff topological space in such a way that multiplication is separately continuous, i.e., the maps \( s \to ts \) and \( s \to st \) from \( S \) into \( S \) are continuous for all \( t \in S \). \( C(S, X) \) denotes the Banach space of all continuous members of \( B(S, X) \).

**Example 2.2.** \( C(S, X) \) is introverted if \( S \) is a compact semitopological semigroup.

For \( \mu \in M(C(S, X)) \) and \( f \in C(S, X) \), we must show that \( T \mu f \) and \( U \mu f \) are continuous.

Let \( g \in C(S) \) and let \( x \in X \). \( g(\cdot)x \in C(S, X) \). Theorem 1.7 implies that \( \mu(g(\cdot)x) = \varphi_{\mu}(g)x \) and \( T \mu (g(\cdot)x) = T \varphi_{\mu}(g)x \). Therefore \( T \mu (g(\cdot)x) \in C(S, X) \) since \( T \varphi_{\mu}(g) \in C(S) \) [1, 2.2.5]. Note the fact that \( C(S, X) = \overline{\text{sp}}\{g(\cdot)x : g \in C(S), \ x \in X\} \) since \( S \) is compact. For \( \epsilon > 0 \) there is \( p(\cdot) = \sum_{i=1}^{n} f_i(\cdot)x_i \), where \( f_i \in C(S) \) and \( x_i \in X \), \( i = 1, 2, \ldots, n \), such that

\[
\|f - p\| < \epsilon.
\]

Now \( p \in C(S, X) \) and

\[
\|T \mu f - T \mu p\| = \max_{s \in S} \|\mu(L_s (f - p))\| \leq \|f - p\| < \epsilon.
\]

Therefore \( T \mu f \in C(S, X) \).

Similarly \( U \mu f \in C(S, X) \). The proof is finished.
Proposition 2.3. Let $A$ be a translation invariant linear subspace of $B(S, X)$ containing the constant functions and let $e : S \to M(A)$ be the evaluation mapping. Then

1. for each $\mu \in M(A)$, $T_\mu : A \to B(S, X)$ is a bounded linear transformation with $\|T_\mu\| \leq \|\mu\|$;
2. the mapping $\mu \to T_\mu : M(A) \to L(A, B(S, X))$ is a bounded transformation;
3. if $\mu \in M(A)$, then $T_\mu(x) = x$, $x \in X$;
4. for all $s \in S$ and $\mu \in M(A)$
   \[ T_\mu L_s = L_s T_\mu \]
   \[ T_\mu R_s = T_{R_s^*} \mu \]
   \[ T_{e(s)} = R_s, \]
   where $R_s^* : M(A) \to M(A)$ is the adjoint of $R_s$;
5. if $f \in A$, then $\{T_\mu f : \mu \in M(A)\}$ is the closure in $B(S, X)$ of $\text{co}(R_s f)$ in the topology of pointwise convergence on $S$.

The proof of the proposition above is like that for [1, 2.2.3], so we omit it.

Definition 2.4. Let $A$ be a translation invariant linear subspace of $B(S, X)$ containing the constant functions, and define

\[ Z_T = \{ \nu \in L(A, X) : T_\nu A \subset A \} \]
and

\[ Z_U = \{ \mu \in L(A, X) : U_\mu A \subset A \}. \]

If $\mu \in L(A, X)$ and $\nu \in Z_T$, define $\mu \nu : A \to X$ by

\[ (\mu \nu)(f) = \mu(T_\nu f) \quad (f \in A). \]

If $\mu \in Z_U$ and $\nu \in L(A, X)$, define $\mu * \nu : A \to X$ by

\[ (\mu * \nu)(f) = \nu(U_\mu f) \quad (f \in A). \]

Definition 2.5. An admissible subspace $A$ of $B(S, X)$ is a norm closed, translation invariant, left introverted subspace of $B(S, X)$ containing the constant functions. In the case that $X = C$, an admissible subspace $A \subset B(S)$ is also required to be conjugate closed.

Let $S$ be a semigroup. Define $\rho_t : S \to S$ and $\lambda_t : S \to S$ by

\[ \rho_t = s t, \quad \lambda_t = t s \quad (s \in S). \]

$S$ is called a right topological semigroup if it is a topological space and $\rho_t$ is continuous for all $t \in S$. Set

\[ \Lambda(S) = \{ s \in S : \lambda_s \text{ is continuous} \}. \]

An affine semigroup $S$ is a semigroup and a convex subset of a vector space in such a way that $\rho_t$ and $\lambda_t$ are affine mappings for each $t \in S$. The requirement that $\rho_t$ and $\lambda_t$ be affine means that if $r, s \in S$ and $a, b \in [0, 1]$ with $a + b = 1$ then

\[(ar + bs)t = atr + bat \text{ and } t(ar + ba) = atr + bts, \]
where (+) denotes vector addition.

The following lemma summarizes the properties of the operation \((\mu, \nu) \to \mu \nu\). The proof is similar to that of [1, 2.2.9]. We omit the statements of the corresponding properties of the operation \((\mu, \nu) \to \mu \ast \nu\).

**Lemma 2.6.** Let \(A\) be as in Definition 2.4 and let \(\epsilon : A \to X\) be the evaluation mapping. Then

1. \(Z_T\) is a linear subspace of \(\mathcal{L}(A, X)\) containing \(\epsilon(S)\);
2. \(\mu \nu \in \mathcal{L}(A, X)\) for all \(\mu \in \mathcal{L}(A, X)\) and \(\nu \in Z_T\);
3. if \(\mu \in \mathcal{L}(A, X), \nu \in Z_T\) and \(s \in S\), we have
   \[T_{\mu \nu} = T_\mu \circ T_\nu,\]
   \[\epsilon(s) \nu = L_s^* \nu,\]
   \[\mu \epsilon(s) = R_s^* \mu,\]
   and
   \[\|\mu \nu\| \leq \|\mu\| \|\nu\|,\]

   where \(L_s^* : M(A) \to M(A)\) is the adjoint of \(L_s\);
4. \(Z_T\) is a right topological semigroup.

The following result is essentially a consequence of the preceding lemma and Propositions 1.5 and 1.6.

**Theorem 2.7.**

1. If \(A\) is an admissible subspace of \(B(S, X)\), then for \(\tau_s\) or \(\tau_w\), and multiplication \((\mu, \nu) \to \mu \nu\), \(M(A)\) is a right topological affine subsemigroup of \(\mathcal{L}(A, X)\), \(\co(\epsilon(S)) \subset \Lambda(M(A))\) and \(\epsilon : S \to M(A)\) is a homomorphism.
2. If we also assume that \(f(S)\) is (weakly) relatively compact for all \(f \in A\), then \(M(A)\) is also compact for \((\tau_w, \tau_s)\).

Let \(S\) be a compact semitopological semigroup. By Example 2.2, \(C(S, X)\) is introverted. Hence \(\mu \nu, \mu \ast \nu \in M(C(S, X))\); indeed, they are equal.

**Proposition 2.8.** Let \(S\) be a compact semitopological semigroup and let \(A = C(S, X)\). Then

1. \(\mu \nu = \mu \ast \nu\) for all \(\mu, \nu \in M(A)\);
2. for \(\tau_s\) and multiplication \((\mu, \nu) \to \mu \nu\), \(M(A)\) is a compact semitopological affine semigroup;
3. if \(S\) is also a topological semigroup, so is \(M(A)\) in \(\tau_s\).

**Proof.** (1). Note that \(\varphi_{\mu} \varphi_{\nu} = \varphi_{\mu \ast \nu}\) [1, 2.2.12 (a)]. Similar to the proof of Example 2.2, we have, for \(g \in C(S)\) and \(x \in X\),

\[\mu \nu(g(\cdot) x) = \mu(T_\nu g(\cdot) x) = \varphi_{\mu}(T_\nu g) x = \varphi_{\mu} \varphi_{\nu}(g) x = \varphi_{\mu \ast \nu}(g) x = \mu \ast \nu(g(\cdot) x).\]
Therefore
\[ \mu \nu(f) = \mu \ast \nu(f) \quad (f \in C(S, X)), \]
i.e., \( \mu \nu = \mu \ast \nu \).

(2) is a consequence of (1) and Theorem 2.7 (1).

To verify (3), we need to show that if \( \mu_\alpha \to \mu \) and \( \nu_\alpha \to \nu \) for \( \tau_\alpha \) then \( \mu_\alpha \nu_\alpha \to \mu \nu \) for \( \tau_\alpha \). Note that \( \varphi_{\mu_\alpha} \varphi_{\nu_\alpha}(g) \to \varphi_\mu \varphi_\nu(g) \) for every \( g \in C(S) \) \([1, 2.2.12 (c)]\). Now, for \( x \in X \),
\[ \mu_\alpha \nu_\alpha(\cdot) = \varphi_{\mu_\alpha} \varphi_{\nu_\alpha}(\cdot) \to \varphi_\mu \varphi_\nu(\cdot) x = \mu_\alpha(\cdot) x. \]

Again using the fact that \( C(S, X) = \overline{\langle g(\cdot) x : g \in C(S), x \in X \rangle} \), we have \( \mu_\alpha \nu_\alpha(f) \to \mu \nu(f) \)
for every \( f \in C(S, X) \).

\section{Invariant Vector-Valued Means}

\( S \) denotes a semigroup which need not have an identity and \( A \) denotes a linear subspace of \( \mathcal{B}(S, X) \) containing the constant functions. Let \( \text{LIM}(A) \) (\( \text{RIM}(A) \)) denotes the set of left (right) invariant means on \( A \). \( A \) is said to be left (right) amenable if \( \text{LIM}(A) \neq \phi \) (\( \text{RIM}(A) \neq \phi \)). If \( A \) is translation invariant, we set
\[ \text{IM}(A) = \text{LIM}(A) \cap \text{RIM}(A) \]
and call members of \( \text{IM}(A) \) invariant means. \( A \) is said to be amenable if \( IM(A) \neq \phi \).

As in the scalar case, we have the following proposition, whose proof is similar to that of \([1, 2.3.5]\); so we omit it.

\textbf{Proposition 3.1.} Let \( A \) be an admissible subspace of \( \mathcal{B}(S, X) \) and let \( \epsilon : S \to \mathcal{L}(A, X) \) be the evaluation mapping.

\begin{enumerate}
\item \( \text{LIM}(A) \) is the set of right zeros of \( M(A) \); hence if \( A \) is left amenable, then \( \text{LIM}(A) \) is a closed ideal of \( M(A) \) contained in every right ideal.
\item If \( A \) is right amenable, then \( \text{RIM}(A) \) is a closed left ideal of \( M(A) \).
\end{enumerate}

\textbf{Corollary 3.2.} Let \( A \) be an admissible subspace of \( \mathcal{B}(S, X) \). If \( A \) is left and right amenable, then it is amenable.

\textit{Proof.} If \( \mu \in \text{LIM}(A) \) and \( \nu \in \text{RIM}(A) \), then \( \mu \nu \in \text{IM}(A) \).

\textbf{Corollary 3.3.} Let \( A \) be an admissible right introverted subspace of \( \mathcal{B}(S, X) \) such that \( \mu \nu = \mu \ast \nu \) for all \( \mu, \nu \in M(A) \). Then \( A \) has at most one invariant mean.

\textit{Proof.} By the proposition and its right introverted analog, if \( \mu, \nu \in \text{IM}(A) \), then \( \nu = \mu \nu = \mu \ast \nu = \mu \).

\textbf{Theorem 3.4.} Let \( A \) be an admissible subspace of \( \mathcal{B}(S, X) \) such that, for each \( f \in A \), the range \( f(S) \) of \( f \) is relatively weakly compact. Let \( K(f) \) denote the closure in \( \mathcal{B}(S, X) \) of \( \text{co}(R_S f) \) for the pointwise topology. The following assertions are equivalent:
(1) $A$ is left amenable;
(2) for each $f \in A$, $K(f)$ contains a constant function;
(3) for each $f \in A$ and $s \in S$, $0 \in K(f - L_s f)$.

Furthermore, if (1) holds then, for each $f \in A$, $\{\mu(f) : \mu \in \text{LIM}(A)\}$ is the set of constant functions in $K(f)$.

Proof. We omit the proofs that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) which do not use weak compactness hypothesis. Here we show that (3) $\Rightarrow$ (1).

For each $f \in A$ and $s \in S$, let

$$M(f, s) = \{\mu \in M(A) : T_\mu(f - L_s f) = 0\}.$$ 

The sets $M(f, s)$ are $\tau_w$-closed, and therefore $\tau_w$-compact. For, let $\{\mu_\alpha\} \subset M(f, s)$ converge to $\mu \in M(A)$. We want to show that $\mu \in M(f, s)$, i.e.,

$$T_\mu(f - L_s f) = 0.$$ 

Note that

$$T_\mu(f - L_s f)(t) = \mu(L_t f - L_{t+s} f) \quad (t \in S)$$

and $\mu_\alpha(L_t f - L_{t+s} f) = T_{\mu_\alpha}(f - L_s f)(t) = 0$ for all $\alpha$. Since $\mu_\alpha(L_t f - L_{t+s} f) \to \mu(L_t f - L_{t+s} f)$ weakly, $\mu(L_t f - L_{t+s} f) = 0$. That is, $T_\mu(f - L_s f) = 0$.

As in the proof of [1, 2.3.11], we can show that the family $\{M(f, s) : f \in A, s \in S\}$ has the finite intersection property. By Proposition 1.6 $M(A)$ is $\tau_w$-compact. So

$$\bigcap\{M(f, s) : f \in A, s \in S\} \neq \emptyset.$$ 

Let $\mu$ be any member of this intersection, then $\mu^2 \in \text{LIM}(A)$.

Let $S$ be a group and let $A$ be a linear subspace of $B(S, X)$. For each $f \in A$ define $\tilde{f} : S \to X$ by

$$\tilde{f}(s) = f(s^{-1}) \quad (s \in S),$$

and set

$$\tilde{A} = \{\tilde{f} : f \in A\}.$$ 

If $\mu \in M(A)$, define $\tilde{\mu} \in M(\tilde{A})$ by

$$\tilde{\mu}(\tilde{f}) = \mu(f) \quad (f \in A).$$

If $\tilde{A} = A$ and $\tilde{\mu} = \mu$, then $\mu$ is said to be inversion invariant.

Theorem 3.5. Let $G$ be a compact Hausdorff topological group. Then $C(G, X)$ has a unique invariant mean $\mu$. Furthermore $\mu$ is inversion invariant.

Proof. The mean $\mu$ can be expressed as

$$\mu(f) = \int_G f d\nu \quad (f \in C(G, X)),$$

where $\nu$ is normalized Haar measure on $G$; the properties of $\mu$ follows from those of $\nu$. 

The scalar version of the next theorem is [1, 2.3.14]; a similar result has appeared in [3], but there $S$ is required to have an identity. A small modification of the proof of [1, 2.3.14] yields a proof of the present theorem.

**Theorem 3.6.** Let $S$ be a compact Hausdorff semitopological semigroup. Then the following assertions hold:

1. $C(S, X)$ is left (respectively right) amenable if and only if $S$ has a unique minimal right (respectively, left) ideal;
2. $C(S, X)$ is amenable if and only if the minimal ideal of $S$ is a compact topological group.

§4. Vector–Valued Weakly Almost Periodic Functions

Let $S$ be a semitopological semigroup; we do not assume $S$ has an identity. Let $WAP(S, X)$ consist of those members $f$ of $C(S, X)$ for which the right orbit $R_S f = \{R_s f : s \in S\}$ is weakly relatively compact in $C(S, X)$.

With a proof similar to that for [1, 4.2.5], one sees that the space $WAP(S, X)$ is a closed translation invariant subspace of $C(S, X)$. When $X = \mathbb{C}$, $WAP(S, X)$ is just $WAP(S)$, the $C^*$-algebra of weakly almost periodic functions on $S$. We note that

$$x^* \circ WAP(S, X) = WAP(S) \quad (x^* \in X^*, \ x^* \neq 0).$$

Recall that $\epsilon : S \to \mathcal{L}(A, X)$ is the evaluation mapping $\epsilon(s)f = f(s), f \in WAP(S, X)$. When $X = \mathbb{C}$ we denote this mapping by $\epsilon'$.

Let $aS^{WAP}$ denote the $w^*$ closure in $WAP(S)^*$ of $coe'(S)$; $aS^{WAP}$ is a compact affine semitopological semigroup [1, 4.2.11].

**Theorem 4.1.** Let $S$ be a semitopological semigroup and let $A = WAP(S, X)$. The following assertions hold:

1. $A$ is an admissible subspace of $B(S, X)$;
2. for $\tau_w$ and multiplication $(\mu, \nu) \to \mu \nu, M(A)$ is an affine semitopological semigroup;
3. if $f(S)$ is weakly relatively compact in $X$ for each $f \in A$, then $M(A)$ is $\tau_w$–compact;
   in this case $A$ is left amenable if and only if $WAP(S)$ is left amenable.

**Proof.** (1) Since $A$ is a closed translation invariant subspace of $C(S, X)$, to show that $A$ is admissible we need to show that $A$ is left introverted, i.e., if $f \in A$ then $T_\mu f \in A$ for all $\mu \in M(A)$.

Define $V : M(A) \to B(S, X)$ by

$$V(\mu) = T_\mu f \quad (\mu \in M(A)).$$

By Proposition 2.3 (5)

$$V(M(A)) = \overline{o}(R_S f), \quad (4.1)$$
the closure being taken in the pointwise topology. Since $f \in A$, $co(R_S f)$ is weakly relatively compact in $A$; in view of (4.1) this implies that $V(M(A))$ is the weak closure in $A$ of $co(R_S f)$. So $T_\mu f \in A$ for all $\mu \in M(A)$.

(2) By Theorem 2.7 (1), for $\tau_w$ and multiplication $(\mu, \nu) \rightarrow \mu \nu$, $M(A)$ is a right topological affine semigroup. It follows from Theorem 1.7 that the mapping $\Pi : \mu \rightarrow \varphi_\mu$ is a $\tau_w - w^*$ homeomorphism of $M(A)$ into $aS^{WAP}$. Since $x^*(\nu(f)) = \varphi_\nu(x^* f)$ for $f \in A$ and $x^* \in X^*$, $x^*(T_\nu f) = T_{\varphi_\nu}(x^* f)$. It follows that $\varphi_{\mu \nu} = \varphi_\mu \varphi_\nu$. Since $\Pi(\mu \nu) = \varphi_{\mu \nu}$, $\Pi$ is a homomorphism too. So $M(A)$ is an affine semitopological semigroup because $aS^{WAP}$ is.

(3) When $A$ satisfies the compactness condition, the $\tau_w$-compactness of $M(A)$ is a consequence of Theorem 2.7 (2). In this case, $M(A) \cong aS^{WAP}$. So we get the last statement.

The proof is complete.

**Remark 4.2.** For $f \in WAP(S, X)$, in general $f(S) \subset X$ is not weakly relatively compact. However, if $S$ admits an identity, it follows from the double limit property (e.g., [2, Theorem 3]) that $f(S)$ is weakly relatively compact. Of course, if $X$ is reflexive then $f(S)$ is weakly relatively compact.

**Theorem 4.3.** For a compact semitopological semigroup $S$, $WAP(S, X) = C(S, X)$.

The theorem holds because the facts of $C(S, X) = \overline{\mathbb{F}}\{f(\cdot) x : f \in C(S), x \in X\}$ and $WAP(S) = C(S)$ [1, 4.2.9].

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