θ -R-CONTINUOUS FUNCTIONS

C.W. BAKER

Department of Mathematics
Indiana University Southeast
New Albany, Indiana 47150

(Received December 4, 1992 and in revised form March 6, 1993)

ABSTRACT. A strong form of continuity, called θ -R-continuity, is introduced. It is shown that θ -R-continuity is stronger than R-continuity. Relationships between θ -R-continuity and various closed graph properties are investigated. Additional properties of θ -R-continuous functions are established.

KEY WORDS AND PHRASES. θ -R-continuity, R-continuity, closed graph property, θ -C-closed, almost-regular.

1980 AMS SUBJECT CLASSIFICATION CODE. 54C10, 54D10.

1. INTRODUCTION.

In this paper a strong form of continuity, which we call θ -R-continuity, is introduced. This form of continuity is stronger than R-continuity developed by Konstadilaki-Savvopoulou and Janković in [1]. As is the case for R-continuity, θ -R-continuity is closely related to regularity and the closed graph property. For example, the range of a θ -R-continuous function is R_1 and a θ -R-continuous function into a T_1 -space satisfies a strong form of the closed graph property. Many of the results in Konstadilaki-Savvopoulou and Janković [1] carry over, with some modification, to θ -R-continuous functions. However, one result in [1] is extended to θ -R-continuous functions.

2. PRELIMINARIES.

The symbols X , Y , and Z represent topological spaces with no separation axioms assumed unless explicitly stated. For a subset A of a space X the closure of A , interior of A , and boundary of A are denoted by $Cl A$, $Int A$, and $Bd A$, respectively. The θ -closure of A , denoted by $Cl_\theta A$, is the set of all x in X such that every closed neighborhood of x intersects A nontrivially. A set A is θ -closed provided that $A = Cl_\theta A$ and a set is θ -open if its complement is

θ -closed. A set A is regular open (resp. regular closed) if $\text{Int Cl } A = A$ (resp. $\text{Cl Int } A = A$). If $\mathcal{F}$ is a filter base on a space X , then by $\text{Cl}_\theta \mathcal{F}$ we mean the filter base $\{\text{Cl}_\theta F : F \in \mathcal{F}\}$.

DEFINITION 1. A function $f: X \rightarrow Y$ is R -continuous (Konstadilaki-Savvopoulou and Jankovic [1]) (resp. strongly θ -continuous (Long and Herrington [2]), weakly continuous (Levine [3])) if for every $x \in X$ and every open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $\text{Cl } f(U) \subseteq V$ (resp. $f(\text{Cl } U) \subseteq V$, $f(U) \subseteq \text{Cl } V$).

3. RELATIONSHIPS TO OTHER CONTINUITY CONDITIONS.

We define a function $f: X \rightarrow Y$ to be θ - R -continuous if for each $x \in X$ and each open subset V of Y containing $f(x)$, there exists an open subset U of X containing x for which $\text{Cl}_\theta f(U) \subseteq V$. The following theorem characterizes θ - R -continuity in terms of filter bases. The proof is straightforward and is omitted.

THEOREM 1. A function $f: X \rightarrow Y$ is θ - R -continuous if and only if for every filter base $\mathcal{F}$ on X , if $\mathcal{F}$ converges to x in X , then the filter base $\text{Cl}_\theta f(\mathcal{F})$ converges to $f(x)$ in Y .

Since for any set A $\text{Cl } A \subseteq \text{Cl}_\theta A$, θ - R -continuity implies R -continuity (and hence implies continuity). The following example shows that the converse implication fails.

EXAMPLE 1. Let $X = \{a, b, c, d\}$, $\mathcal{T} = \{X, \emptyset, \{c\}, \{c, d\}, \{a, b, c\}\}$ and $f: (X, \mathcal{T}) \rightarrow (X, \mathcal{T})$ be the constant map $f(x) = a$ for each $x \in X$. It is easily checked that f is R -continuous but not θ - R -continuous.

Since in a regular space the closure and θ -closure operators agree, obviously if $f: X \rightarrow Y$ is R -continuous and Y is regular, then f is θ - R -continuous. Also because the closure and θ -closure operators agree on open sets, it follows that if $f: X \rightarrow Y$ is R -continuous and open, then f is θ - R -continuous. R -continuity (and hence θ - R -continuity) implies strong θ -continuity (Konstadilaki-Savvopoulou and Jankovic [1]). The following theorem establishes conditions under which this implication can be reversed.

DEFINITION 2. Jankovic and Rose [4]. A function $f: X \rightarrow Y$ is weakly θ -closed provided there exists an open basis $\mathcal{B}$ for the topology on X for which $\text{Cl}_\theta f(U) \subseteq f(\text{Cl } U)$ for every $U \in \mathcal{B}$.

THEOREM 2. If $f: X \rightarrow Y$ is strongly θ -continuous and weakly θ -closed, then f is θ - R -continuous.

PROOF. Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Since f is strongly θ -continuous, there exists an open subset U of X containing x for which $f(\text{Cl } U) \subseteq V$. Let $\mathcal{B}$ be an open basis for the topology on X such that $\text{Cl}_\theta f(W) \subseteq f(\text{Cl } W)$ for every $W \in \mathcal{B}$. We may assume $U \in \mathcal{B}$. Thus $\text{Cl}_\theta f(U) \subseteq f(\text{Cl } U) \subseteq V$ and hence f is θ - R -continuous. $\square$

In Konstadilaki-Savvopoulou and Jankovic [1] it is proved that a continuous function from a locally compact Hausdorff Space into a Hausdorff space is R -continuous. We shall prove that under these conditions the function is actually θ - R -continuous. The essential reason is that a compact subset of a Hausdorff space is θ -closed.

Except for the use of this fact, the proof is the same as in Konstadilaki-Savvopoulou and Jankovic [1].

THEOREM 3. If $f: X \rightarrow Y$ is continuous, X is locally compact Hausdorff, and Y is Hausdorff, then f is θ -R-continuous.

PROOF. Since X is locally compact Hausdorff, X is regular and therefore f is strongly θ -continuous. From the local compactness of X , we obtain an open basis $\mathcal{B}$ for the topology on X consisting of sets with compact closures. Let $U \in \mathcal{B}$. Since f is continuous, $f(\text{Cl } U)$ is compact in Y and hence θ -closed because Y is Hausdorff. Thus $\text{Cl}_{\theta} f(U) \subseteq \text{Cl}_{\theta} f(\text{Cl } U) = f(\text{Cl } U)$. Therefore f is weakly θ -closed and hence by Theorem 2 f is θ -R-continuous. $\square$

4. CLOSED GRAPH PROPERTIES.

From Konstadilaki-Savvopoulou and Jankovic [1] the graph of an R-continuous function into a T_1 -space is θ -closed with respect to the domain. In this section an analogous result is proved for θ -R-continuous functions.

DEFINITION 3. Baker [5]. A subset U of a space X is θ -C-open provided there exists a subset A of X for which $U = X - \text{Cl}_{\theta} A$.

DEFINITION 4. Let $f: X \rightarrow Y$ be a function. The graph of f , denoted by $G(f)$, is θ -C-closed with respect to X if for each $(x, y) \notin G(f)$ there exist subsets U and V of X and Y , respectively, with $x \in U$, $y \in V$, U open, V θ -C-open, and $((\text{Cl } U) \times V) \cap G(f) = \emptyset$ (or equivalently $f(\text{Cl } U) \cap V = \emptyset$).

THEOREM 4. If $f: X \rightarrow Y$ is θ -R-continuous and Y is T_1 , then $G(f)$ is θ -C-closed with respect to X .

PROOF. Assume $(x, y) \notin G(f)$. Since $y \neq f(x)$ and Y is T_1 , there exists an open subset V of Y such that $f(x) \in V$ and $y \notin V$. The θ -R-continuity of f implies the existence of an open subset U of X containing x such that $\text{Cl}_{\theta} f(U) \subseteq V$. Therefore $(x, y) \in (\text{Cl } U) \times (Y - \text{Cl}_{\theta} f(U))$ which is disjoint from $G(f)$ because if $a \in \text{Cl } U$, then $f(a) \in f(\text{Cl } U) \subseteq \text{Cl } f(U) \subseteq \text{Cl}_{\theta} f(U)$. Note that $Y - \text{Cl}_{\theta} f(U)$ is θ -C-open. $\square$

Next we establish conditions under which a weak form of continuity and some type of closed graph property imply θ -R-continuity.

DEFINITION 5. Jankovic and Rose [4]. The graph of a function $f: X \rightarrow Y$ is θ -closed with respect to Y if for each $(x, y) \notin G(f)$, there exist open subsets U and V of X and Y , respectively, with $x \in U$, $y \in V$, and $(U \times \text{Cl } V) \cap G(f) = \emptyset$ (or equivalently $f(U) \cap \text{Cl } V = \emptyset$).

DEFINITION 6. Singal and Arya [6]. A space X is almost-regular if each regular closed set C and each $x \in X - C$ can be separated by disjoint open subsets of X .

The assumption that the codomain is almost-regular is required in several of the following theorems. Note that under this assumption, θ -R-continuity is not equivalent to R-continuity. The space in Example 1 is almost-regular whereas the function is R-continuous but not θ -R-continuous.

From Jankovic and Rose [4] we have that a space X is almost-regular if and only if regular closed sets are θ -closed. This fact is used in the following proof.

DEFINITION 7. Konstadilaki-Savvopoulou and Jankovic [1]. A space X is rim-compact provided there exists an open basis for the topology on X consisting of sets with compact boundaries.

THEOREM 5. If $f: X \rightarrow Y$ is weakly continuous and has a θ -closed graph with respect to Y and Y is rim-compact and almost-regular, then f is θ -R-continuous.

PROOF. Let $x \in X$ and let V be an open subset of X containing $f(x)$. Since Y is rim-compact, there exists an open subset W of Y for which $f(x) \in W \subseteq V$ and $\text{Bd } W$ is compact. The weak continuity of f implies the existence of an open subset U of X containing x such that $f(U) \subseteq \text{Cl } W$.

Let $y \in \text{Bd } W$. Since $f(x) \in W$ and $W \cap \text{Bd } W = \emptyset$, $(x, y) \notin G(f)$. Because $G(f)$ is θ -closed with respect to Y , there exist open subsets A_y and B_y of X and Y , respectively, for which $x \in A_y$, $y \in B_y$, and $f(A_y) \cap \text{Cl } B_y = \emptyset$. It follows that $(\text{Cl}_\theta f(A_y)) \cap B_y = \emptyset$.

The collection $\{B_y : y \in \text{Bd } W\}$ is an open cover of $\text{Bd } W$, which is compact. Hence there is a finite collection $\{B_{y_i} : i = 1, 2, \dots, n\}$ for which $\text{Bd } W \subseteq \bigcup \{B_{y_i} : i = 1, 2, \dots, n\}$. Let $U_0 = U \cap (\bigcap \{A_{y_i} : i = 1, 2, \dots, n\})$. Then $\text{Cl}_\theta f(U_0) \subseteq \text{Cl}_\theta f(\bigcap \{A_{y_i} : i = 1, 2, \dots, n\}) \subseteq \text{Cl}_\theta \bigcap \{f(A_{y_i}) : i = 1, 2, \dots, n\} \subseteq \bigcap \{\text{Cl}_\theta f(A_{y_i}) : i = 1, 2, \dots, n\}$ which is disjoint from $\bigcup \{B_{y_i} : i = 1, 2, \dots, n\}$ and hence disjoint from $\text{Bd } W$. Therefore $(\text{Cl}_\theta f(U_0)) \cap \text{Bd } W = \emptyset$. However, $\text{Cl}_\theta f(U_0) \subseteq \text{Cl}_\theta f(U) \subseteq \text{Cl}_\theta \text{Cl } W = \text{Cl } W$. (This last equality follows from the fact that $\text{Cl } W$ is regular closed and Y is almost-regular.) Thus we have that $\text{Cl}_\theta f(U_0) \subseteq (\text{Cl } W) - \text{Bd } W \subseteq W \subseteq V$ and therefore f is θ -R-continuous. $\square$

5. ADDITIONAL PROPERTIES.

In Konstadilaki-Savvopoulou and Jankovic [1] it is proved that a function $f: X \rightarrow Y$ is R-continuous if and only if for every $x \in X$ and every closed subset F of Y with $f(x) \notin F$, there exist open subsets U and V of X and Y , respectively, such that $x \in U$, $F \subseteq V$, and $f(U) \cap V = \emptyset$. The following three theorems are analogous results for θ -R-continuous functions.

THEOREM 6. If $f: X \rightarrow Y$ is θ -R-continuous, then for every $x \in X$ and every closed subset F of Y such that $f(x) \notin F$, there exists an open subset U of X containing x and a θ -C-open subset V of Y with $F \subseteq V$ such that $f(\text{Cl } U) \cap V = \emptyset$.

PROOF. Let $x \in X$ and let F be a closed subset of Y with $f(x) \in X - F$. There exists an open subset U of X containing x such that $\text{Cl}_\theta f(U) \subseteq Y - F$. Let $V = Y - \text{Cl}_\theta f(U)$. Then V is θ -C-open and $F \subseteq V$. Since f is continuous, $f(\text{Cl } U) \subseteq \text{Cl } f(U) \subseteq \text{Cl}_\theta f(U)$. Therefore $f(\text{Cl } U) \cap V = \emptyset$. $\square$

If the condition that V be θ -C-open is replaced with the stronger requirement that V be θ -open, then the implication in Theorem 6 can be reversed.

THEOREM 7. Let $f: X \rightarrow Y$ be a function. If for every $x \in X$ and every closed subset F of Y with $f(x) \notin F$ there exists an open subset U of X containing x and a θ -open subset V of Y with $F \subseteq V$ such that $f(U) \cap V = \emptyset$, then f is θ -R-continuous.

PROOF. Let $x \in X$ and let V be an open subset of Y with $f(x) \in V$. Let $F = Y - V$. Since $f(x) \notin F$, there exists an open subset U of X containing x and a θ -open subset W of Y with $F \subseteq W$ and $f(U) \cap W = \emptyset$. Then $f(U) \subseteq Y - W$. Thus $\text{Cl}_\theta f(U) \subseteq \text{Cl}_\theta (Y - W) = Y - W \subseteq Y - F = V$. Therefore f is θ -R-continuous. $\square$

In an almost-regular space $\text{Cl}_\theta \text{Cl}_\theta A = \text{Cl}_\theta A$ for any set A (Jankovic and Rose [4]). It follows that in an almost-regular space θ -C-openness is equivalent to θ -openness. Therefore we have the following result.

THEOREM 8. Let X and Y be topological spaces with Y almost-regular. Then $f: X \rightarrow Y$ is θ -R-continuous if and only if for every $x \in X$ and every closed subset F of Y with $f(x) \notin F$, there exist an open subset U of X containing x and an θ -C-open subset V of Y with $F \subseteq V$ such that $f(U) \cap V = \emptyset$.

In the definition of R-continuity the condition " $\text{Cl } f(U) \subseteq V$ " can be replaced with " $\text{Cl } f(\text{Cl } U) \subseteq V$ " (Konstadilaki-Savvopoulou and Jankovic [1]). If we require the codomain to be almost-regular, the following similar result holds for θ -R-continuity.

THEOREM 9. Let X and Y be topological spaces with Y almost-regular. Then $f: X \rightarrow Y$ is θ -R-continuous if and only if for each $x \in X$ and each open subset V of Y containing $f(x)$, there exists an open subset U of X containing x such that $\text{Cl}_\theta f(\text{Cl } U) \subseteq V$.

PROOF. Assume $f: X \rightarrow Y$ is θ -R-continuous. Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Then there exists an open subset U of X containing x such that $\text{Cl}_\theta f(U) \subseteq V$. Since f is continuous, $\text{Cl}_\theta f(\text{Cl } U) \subseteq \text{Cl}_\theta \text{Cl } f(U) \subseteq \text{Cl}_\theta \text{Cl}_\theta f(U)$ and since Y is almost-regular, $\text{Cl}_\theta \text{Cl}_\theta f(U) = \text{Cl}_\theta f(U) \subseteq V$. Thus $\text{Cl}_\theta f(\text{Cl } U) \subseteq V$.

The converse implication is immediate. $\square$

In Konstadilaki-Savvopoulou and Jankovic [1] the range of an R-continuous function is shown to be R_0 . Here we show that the range of a θ -R-continuous function satisfies the stronger R_1 condition. The following definition is implicit in Theorem 3.1 in Jankovic [7].

DEFINITION 8. Jankovic [7]. A space X is R_1 if for each open set U and each $x \in U$, $\text{Cl}_\theta \{x\} \subseteq U$.

THEOREM 10. If $f: X \rightarrow Y$ is a θ -R-continuous surjection, then Y is R_1 .

PROOF. Let $y \in Y$ and let V be an open subset of Y containing y . Let $x \in X$ such that $y = f(x)$. Since f is θ -R-continuous, there exists an open subset U of X containing x for which $\text{Cl}_\theta f(U) \subseteq V$. Then $\text{Cl}_\theta \{y\} \subseteq \text{Cl}_\theta f(U) \subseteq V$. $\square$

We close this section with a sample of the basic properties of θ -R-continuous functions concerning composition and restriction. Most of these are analogues of the corresponding properties for R-continuous or continuous functions. The proofs are straightforward and are omitted.

THEOREM 11. If $f: X \rightarrow Y$ is continuous and $g: Y \rightarrow Z$ is θ -R-continuous, then $g \circ f: X \rightarrow Z$ is θ -R-continuous.

COROLLARY 1. If both $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are θ -R-continuous, then $g \circ f: X \rightarrow Z$ is θ -R-continuous.

THEOREM 12. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are functions with $g \circ f: X \rightarrow Z$ θ -R-continuous, and f is an open surjection, then g is θ -R-continuous

THEOREM 13. If $f: X \rightarrow Y$ is θ -R-continuous, $A \subseteq X$, and $f(A) \subseteq B \subseteq Y$, then $f|_A: A \rightarrow B$ is θ -R-continuous.

REFERENCES

1. Konstadilaki-Savvopoulou, Ch. and Jankovic, D., R-continuous functions, Internat. J. Math. & Math. Sci. **15** (1992), 57-64.
2. Long, P.E. and Herrington, L.L., Strongly θ -continuous functions, J. Korean Math Soc. **18** (1961), 21-28.
3. Levine, N., A decomposition of continuity in topological spaces, Amer. Math. Monthly **68** (1961), 44-46.
4. Jankovic, D.S. and Rose, D.A., Weakly closed functions, manuscript.
5. Baker, C.W., On θ -C-open sets, Internat. J. Math. & Math. Sci. **15** (1992), 255-260.
6. Singal M.K. and Arya, S.P., On almost-regular spaces, Glas. Mat. Ser. III **24** (1969), 89-99.
7. Jankovic D., On some separation axioms and θ -closure, Mat. Vernik **17** (1980), 439-449.