A RESULT OF COMMUTATIVITY OF RINGS

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Abstract. In this paper we prove the following:

THEOREM. Let $n > 1$ and m be fixed relatively prime positive integers and k is any non-negative integer. If R is a ring with unity 1 satisfying $x^k[x^n, y] = [x, y^m]$ for all $x, y \in R$ then R is commutative.

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1. INTRODUCTION.

Psomopoulos [12] proved that if R is a ring with unity satisfying the properties that for each $x, y \in R$,

- (i) $x^k[x^n, y] = [x, y^m]$
- (ii) $(xy)^n = x^n y^n$
- (iii) $(xy)^k = x^k y^k$

where $n > 1$ and m are fixed relatively prime positive integers and k is any non-negative integer, then R is commutative. In this paper we prove the theorem stated in the abstract which improve above theorem of Psomopolous [12] where conditions (ii) and (iii) are superfluous.

Throughout, R will denote an associative ring with unit 1. We use the following notations.

$Z(R)$, the center of R .

$[x, y] = xy - yx$

$C(R)$, the commutator ideal of R .

$N(R)$, the set of all nilpotent elements of R .

$D(R)$, the set of all zero divisors in R .

2. MAIN RESULTS.

We state our main result as follows.

MAIN THEOREM. Let $n > 1$ and m be fixed relatively prime positive integers and k is any non-negative integer. If R is a ring with unity 1 satisfying

$$(*) \quad x^k[x^n, y] = [x, y^m] \quad \text{for all } x, y \in R$$

then R is commutative.

We begin with the following lemmas which will be used in proving our main theorem.

LEMMA 1 ([2], Theorem 1). Let R be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial identity in non-commuting in-determinates, its coefficient being integers with highest common factor one. If there exists no prime p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(X) = 0$, then R has a nil commutator ideal and the nilpotent elements of R form an ideal.

LEMMA 2 ([8], p. 221). If $x, y \in R$ and $[x, y]$ commute with x , then $[x^n, y] = nx^{n-1}[x, y]$ for all positive integer n .

LEMMA 3 ([9]). Let R be a ring with unity and let $f: R \rightarrow R$ be a function such that $f(x+1) = f(x)$ for all $x \in R$. If for some positive integer n , $x^n f(x) = 0$ for all x in R , then necessarily $f(x) = 0$.

LEMMA 4. If R is a ring satisfying (*) in the hypothesis of the main theorem then

$$C(R) \subseteq N(R) \subseteq Z(R)$$

PROOF. By Lemma 3 of [12] we have $N(R) \subseteq Z(R)$ when R satisfies $x^k[x^n, y] = [x, y^m]$ for all $x, y \in R$. This is a polynomial identity with coprime integral coefficients. But if we consider (i) $x = e_{22}$ and $y = e_{21}$, if $n > 1, m > 1$ and (ii) $x = e_{21}$ and $y = e_{22}$ if $n > 1$ and $m = 1$, we find that no ring of 2×2 matrices over $GF(p)$, p a prime, satisfies this identity. Hence by Lemma 1, $C(R)$ is a nil ideal and thus

$$C(R) \subseteq N(R) \subseteq Z(R).$$

PROOF OF MAIN THEOREM. By Lemma 4, we have

$$C(R) \subseteq N(R) \subseteq Z(R)$$

Thus all commutators are central. Moreover, we know that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_α each of which a homomorphic image of R satisfies the hypotheses of the theorem. Thus we can assume that R is subdirectly irreducible ring. Hence I , the intersection of all non-zero ideals is non-zero.

CASE 1. Let $n > 1$ and $m > 1$.

By using Lemma 2, we write (*) as

$$nx^{n+k-1}[x, y] = [x, y^m] \quad \text{for all } x, y \in R. \quad (2.1)$$

Let $c = 2^{n+k} - 2 > 0$, then

$$\begin{aligned} ncx^{n+k-1}[x, y] &= n\{2^{n+k}x^{n+k-1}[x, y] - 2x^{n+k-1}[x, y]\} \\ &= n2^{n+k}x^{n+k-1}[x, y] - 2nx^{n+k-1}[x, y] \\ &= n(2x)^{n+k-1}[2x, y] - 2[x, y^m] \\ &= [2x, y^m] - 2[x, y^m] = 0. \end{aligned} \quad (2.2)$$

Hence $ncx^{n+k-1}[x, y] = 0$ for all $x, y \in R$. Now replace x by $x+1$ and by using Lemma 3, we get

$$nc[x, y] = 0. \quad (2.3)$$

All cummutators are central and hence by Lemma 2

$$[x^{nc}, y] = ncx^{nc-1}[x, y] = 0.$$

Thus $x^{nc} \in Z(R)$ for all $x \in R$. We replace y by y^m in (2.1) to get

$$nx^{n+k-1}[x, y^m] = [x, (y^m)^m]. \quad (2.4)$$

Thus

$$\begin{aligned}
 nx^{n+k-1}[x, y^m] &= n[x, y^m]x^{n+k-1} \\
 &= nmy^{m-1}[x, y]x^{n+k-1} \\
 &= nmy^{m-1}x^{n+k-1}[x, y] \\
 &= my^{m-1}[x, y^m]
 \end{aligned} \tag{2.5}$$

and

$$[x, (y^m)^m] = m(y^m)^{m-1}[x, y^m] = my^{m-1}y^{(m-1)^2}[x, y^m]. \tag{2.6}$$

Thus by using (2.5) and (2.6), we can write (2.4) as

$$\begin{aligned}
 my^{m-1}[x, y^m] &= my^{m-1}y^{(m-1)^2}[x, y^m] \\
 my^{m-1}(1 - y^{(m-1)^2})[x, y^m] &= 0
 \end{aligned}$$

Hence

$$my^{m-1}(1 - y^{nc(m-1)^2})[x, y^m] = 0. \tag{2.7}$$

We claim that

$$D(R) \subseteq Z(R).$$

Let $a \in D(R)$ then

$$a^{nc(m-1)^2} \in Z(R) \cap D(R) \quad \text{and} \quad Ia^{nc(m-1)^2} = 0.$$

By (2.7), we get

$$ma^{m-1}(1 - a^{nc(m-1)^2})[x, a^m] = 0.$$

Thus

$$(1 - a^{nc(m-1)^2})ma^{m-1}[x, a^m] = 0. \tag{2.8}$$

If $ma^{m-1}[x, a^m] \neq 0$, then

$$1 - a^{nc(m-1)^2} \in D(R)$$

Hence $I(1 - a^{nc(m-1)^2}) = 0$ and $I = 0$. This is contradiction. Now we have

$$ma^{m-1}[x, a^m] = 0. \tag{2.9}$$

Thus

$$\begin{aligned}
 n^2x^{n+k-1}x^{n+k-1}[x, a] &= nx^{n+k-1}[x, a^m] \\
 &= [x, (a^m)^m] \\
 &= m(a^m)^{m-1}[x, a^m] \\
 &= a^{(m-1)^2}ma^{m-1}[x, a^m] = 0.
 \end{aligned} \tag{2.10}$$

Replacing x by $x + 1$ in (2.10) and using Lemma 3 we get

$$n^2[x, a] = 0. \tag{2.11}$$

By using Lemma 2, we can write (*) as

$$x^k[x^a, y] = my^{m-1}[x, y]. \tag{2.12}$$

Let $d = 2^m - 2 > 0$. Then

$$\begin{aligned}
m d y^{m-1}[x, y] &= m 2^m y^{m-1}[x, y] - 2 y^{m-1}[x, y] \\
&= m (2y)^{m-1}[x, 2y] - 2 m y^{m-1}[x, y] \\
&= x^k[x^n, 2y] - 2x^k[x^n, y] \\
&= x^k[x^n, 2y] - x^k[x^n, 2y] = 0.
\end{aligned} \tag{2.13}$$

Hence $m d y^{m-1}[x, y] = 0$ for all $x, y \in R$. Now replacing y by $y + 1$ and by using Lemma 3, we get

$$m d[x, y] = 0. \tag{2.14}$$

All commutators are central and hence by Lemma 2

$$[x, y^{md}] = m d y^{md-1}[x, y] = 0$$

Thus $y^{md} \in Z(R)$ for all $y \in R$. Now replacing x by x^n in (2.12), we get

$$x^{nk}[(x^n)^n, y] = m y^{m-1}[x^n, y] \tag{2.15}$$

Thus

$$\begin{aligned}
x^{nk}[(x^n)^n, y] &= x^{nk} n (x^n)^{n-1} [x^n, y] \\
&= n x^{nk} x^{n-1} x^{(n-1)^2} [x^n, y] \\
&= n x^{n+k-1} x^{nk-k} x^{(n-1)^2} [x^n, y] \\
&= n x^{n+k-1} x^{(n-1)k} x^{(n-1)^2} [x^n, y] \\
&= n x^{n+k-1} x^{(n-1)(n+k-1)} [x^n, y]
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
m y^{m-1}[x^n, y] &= m [x^n, y] y^{m-1} \\
&= m n x^{n-1} [x, y] y^{m-1} \\
&= m n x^{n-1} y^{m-1} [x, y] \\
&= n x^{n-1} m y^{m-1} [x, y] \\
&= n x^{m-1} x^k [x^n, y] \\
&= n x^{n+k-1} [x^n, y]
\end{aligned} \tag{2.17}$$

Thus by using (2.16) and (2.17) we can write (2.15) as

$$n x^{n+k-1} x^{(n-1)(n+k-1)} [x^n, y] = n x^{n+k-1} [x^n, y].$$

$$n x^{n+k-1} (1 - x^{(n-1)(n+k-1)}) [x^n, y] = 0. \tag{2.18}$$

Hence by using (2.18) we get,

$$n x^{n+k-1} (1 - x^{md(n-1)(n+k-1)}) [x^n, y] = 0. \tag{2.19}$$

Since $a \in D(R)$, we have

$$a^{md(n-1)(n+k-1)} \in Z(R) \cap D(R) \quad \text{and} \quad I a^{md(n-1)(n+k-1)} = 0.$$

By (2.19) we get

$$n a^{n+k-1} (1 - a^{md(n-1)(n+k-1)}) [a^n, y] = 0.$$

This can be written as

$$(1 - a^{md(n-1)(n+k-1)}) n a^{n+k-1} [a^n, y] = 0. \tag{2.20}$$

If $n a^{n+k-1} [a^n, y] \neq 0$. Then

$$1 - a^{md(n-1)(n+k-1)} \in D(R)$$

and $I(1 - a^{md(n-1)(n+k-1)}) = 0$ and hence $I = 0$, which is a contradiction. Thus we have

$$na^{n+k-1}[a^n, y] = 0. \quad (2.21)$$

Now

$$\begin{aligned} m^2 y^{m-1} y^{m-1} [a, y] &= m y^{m-1} [a, y] m y^{m-1} - a^k [a^n, y] m y^{m-1} \\ &= a^k m y^{m-1} [a^n, y] - a^k a^{nk} [(a^n)^n, y] \\ &= a^{nk+k} n (a^n)^{n-1} [a^n, y] - a^{nk+k} n a^{n-1} a^{(n-1)^2} [a^n, y] \\ &= a^{nk} a^{(n-1)^2} n a^{n+k-1} [a^n, y] = 0. \end{aligned} \quad (2.22)$$

Replacing y by $y+1$ in (2.22) and using Lemma 3, we get

$$m^2 [a, y] = 0 \quad \text{for all } y \in R.$$

Replacing y by x , we get

$$m^2 [x, a] = 0 \quad \text{for all } x \in R. \quad (2.23)$$

But m^2 and n^2 are relatively prime. Hence there exists integers α and β such that $m^2\alpha + n^2\beta = 1$. Multiplying (2.11) by β and (2.23) by α and adding, we get

$$[x, a] = 0 \quad \text{for all } x \in R.$$

Hence $a \in Z(R)$, which proves our claim.

We know that x^{nc} and $x^{ncm} \in Z(R)$. Thus

$$\begin{aligned} (x^{nc} - x^{ncm}) n x^{n+k-1} [x, y] &= n x^{nc} x^{n+k-1} [x, y] - n x^{ncm} x^{n+k-1} [x, y] \\ &= n x^{n+k-1} [x, x^{nc} y] - x^{ncm} [x, y^m] \\ &= n x^{n+k-1} [x, x^{nc} y] - [x, (x^{nc} y)^m] \\ &= n x^{n+k-1} [x, x^{nc} y] - n x^{n+k-1} [x, x^{nc} y] = 0. \end{aligned}$$

Thus $(x - x^{ncm-nc+1}) n x^{n+k-1} x^{nc-1} [x, y] = 0$, i.e.

$$n(x - x') x^p [x, y] = 0 \quad \text{for all } x, y \in R \quad (2.24)$$

where $t = ncm - nc + 1 > 1$ and $p = n + k + nc - 2$.

We know that y^{md} and $y^{mdn} \in Z(R)$. Thus

$$\begin{aligned} (y^{md} - y^{mdn}) m y^{m-1} [x, y] &= m y^{md} y^{m-1} [x, y] - m y^{mdn} y^{m-1} [x, y] \\ &= m y^{m-1} [xy^{md}, y] - y^{mdn} x^k [x^n, y] \\ &= m y^{m-1} [xy^{md}, y] - x^k [(xy^{md})^n, y] \\ &= m y^{m-1} [xy^{md}, y] - m y^{m-1} [xy^{md}, y] = 0 \end{aligned}$$

Thus $m(y - y^{mdn-md+1}) y^{md-1} y^{m-1} [x, y] = 0$. That is $m(y - y^n) y^q [x, y] = 0$ for all $x, y \in R$, where $u = mdn - md + 1 > 1$ and $q = md + m - 2$. Interchanging x and y , we get

$$m(x - x^n) x^q [x, y] = 0 \quad \text{for all } x, y \in R. \quad (2.25)$$

We know that $(m, n) = 1$. Hence there exists integers α and β such that $m\alpha + n\beta = 1$. Multiplying (2.24) by $\beta(x - x^n) x^q$ and multiplying (2.25) by $\alpha(x - x') x^p$ and adding, we get

$$(x - x')(x - x^n) x^{p+q} [x, y] = 0 \quad \text{for all } x, y \in R$$

This can be written as

$$(x - x^2h(x))x^{p+q+1}[x, y] = 0 \quad \text{for all } x, y \in R \quad (2.26)$$

where $h(x)$ is a polynomial in x with integers coefficients.

Suppose R is not commutative. Then by a well known result of Herstein [6], there exists $x \in R$ such that $x - x^2h(x) \notin Z(R)$. From this it is clear that $x \notin Z(R)$. Hence x and $x - x^2h(x)$ is not a zero divisor. Hence $(x - x^2h(x))x^{p+q+1}$ is also not a zero divisor. Thus

$$[x, y] = 0 \quad \text{for all } y \in R \quad (2.27)$$

This gives a contradiction. Hence R is commutative.

CASE 2: Let $n > 1$ and $m = 1$. Then (*) can be written as

$$x^k[x^n, y] = [x, y] \quad (2.28)$$

Let $e = 2^{k+n} - 2 > 0$. Then

$$\begin{aligned} e[x, y] &= 2^{k+n}[x, y] - 2[x, y] \\ &= 2^{k+n}x^k[x^n, y] - [2x, y] \\ &= (2x)^k[(2x)^n, y] - [2x, y] \\ &= [2x, y] - [2x, y] = 0. \end{aligned}$$

All commutators are central and hence by Lemma 2,

$$[x^e, y] = ex^{e-1}[x, y] = 0 \quad \text{for all } x, y \in R.$$

Hence $e^e \in Z(R)$. Now replacing x by x^n in (2.28) we get

$$x^{nk}[(x^n)^n, y] = [x^n, y]. \quad (2.29)$$

Thus

$$\begin{aligned} x^{nk}[(x^n)^n, y] &= nx^{nk}(x^n)^{n-1}[x^n, y] \\ &= nx^{nk}x^{(n-1)}x^{(n-1)^2}[x^n, y] \\ &= nx^{nk-k}x^{n+k-1}x^{(n-1)^2}[x^n, y] \\ &= nx^{n+k-1}x^{(n-1)(n+k-1)}[x^n, y] \\ &= nx^{n-1}x^{(n-1)(n+k-1)}x^k[x^n, y] \\ &= nx^{n-1}x^{(n-1)(n+k-1)}[x, y]. \end{aligned} \quad (2.30)$$

and

$$[x^n, y] = nx^{n-1}[x, y]. \quad (2.31)$$

Thus, by using (2.30) and (2.31), we can write (2.29) as

$$nx^{n-1}x^{(n-1)(n+k-1)}[x, y] = nx^{n-1}[x, y].$$

Thus

$$nx^{n-1}(1 - x^{(n-1)(n+k-1)})[x, y] = 0. \quad (2.32)$$

Thus, by using (2.32), we get

$$nx^{n-1}(1 - x^{e(n-1)(n+k-1)})[x, y] = 0 \quad (2.33)$$

Let $a \in D(R)$ then

$$a^{e(n-1)(n+k-1)} \in Z(R) \cap D(R) \quad \text{and} \quad Ia^{e(n-1)(n+k-1)} = 0.$$

By using (2.33) we get

$$na^{n-1}(1-a^{e(n-1)(n+k-1)})[a, y] = 0.$$

Then

$$(1-a^{e(n-1)(n+k-1)})na^{n-1}[a, y] = 0. \quad (2.34)$$

If $na^{n-1}[a, y] \neq 0$. Then

$$(1-a^{e(n-1)(n+k-1)}) \in D(R)$$

and $I(1-a^{e(n-1)(n+k-1)}) = 0$. Hence $I = 0$, which is a contradiction. Thus we have

$$[a^n, y] = na^{n-1}[a, y] = 0.$$

Hence $a^k[a^n, y] = [a, y] = 0$ for all $y \in R$. Now $a \in Z(R)$. We know that x^e and $x^{en} \in Z(R)$. Thus

$$\begin{aligned} (x^e - x^{en+ek})[x, y] &= x^e[x, y] - x^{en+ek}[x, y] \\ &= [x^{e+1}, y] - x^{en+ek}x^k[x^n, y] \\ &= [x^{e+1}, y] - x^{ek}x^k[(x^{e+1})^n, y] \\ &= [x^{e+1}, y] - x^{(e+1)k}[(x^{e+1})^n, y] \\ &= [x^{e+1}, y] - [x^{e+1}, y] = 0. \end{aligned}$$

Hence $(x - x^{en+ek-e+1})x^{e-1}[x, y] = 0$. If R is not commutative then by a well known result of Herstein [5] there exists $x \in R$ such that $x - x^v \notin Z(R)$ where $v = en + ek - e + 1 > 1$. By using smaller arguments as in the last paragraph of case 1, we get a contradiction. Hence R is commutative.

We give examples which show that all the hypotheses of our main theorem are essential. The following example show that R is not commutative if m and n are not relatively prime or the ring is without unity in the hypothesis of our main theorem.

EXAMPLE 1. Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in F, F : \text{field} \right\}$$

Then R is a ring without unity satisfying $x^k[x^2, y] = [x, y^3]$ and for all non-negative integer k . But R is not commutative.

EXAMPLE 2. Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in GF(2) \right\}$$

Then R is a ring with unity satisfying $x^k[x^4, y] = [x, y^4]$ for all $x, y \in R$ and for all non-negative integer k . But R is not commutative.

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