FINITE COMPLETELY PRIMARY RINGS IN WHICH THE PRODUCT OF ANY TWO ZERO DIVISORS OF A RING IS IN ITS COEFFICIENT SUBRING

YOUSIF ALKHAMEES

Department of Mathematics
King Saud University
P.O. Box 2455
Riyadh 11451, Saudi Arabia

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ABSTRACT. According to general terminology, a ring R is completely primary if its set of zero divisors J forms an ideal. Let R be a finite completely primary ring. It is easy to establish that J is the unique maximal ideal of R and R has a coefficient subring S (i.e. R/J isomorphic to S/pS) which is a Galois ring. In this paper we give the construction of finite completely primary rings in which the product of any two zero divisors is in S and determine their enumeration. We also show that finite rings in which the product of any two zero divisors is a power of a fixed prime p are completely primary rings with either $J^2=0$ or their coefficient subring is Z_{p^n} with $n=2$ or 3 . A special case of these rings is the class of finite rings, studied in [2], in which the product of any two zero divisors is zero.

KEY WORDS AND PHRASES. Finite completely primary ring, Galois ring.

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1. INTRODUCTION.

All rings considered in this paper are associative with identity. Let R be a finite completely primary ring. It is easy to see (cf. [5]) that $|R|=p^{mr}$, $|J|=p^{(m-1)r}$, and the characteristic of R is p^n , for some prime p and positive integers m, n and r with $1 \leq n \leq m$. If $n=m$, then R is of the form $Z_{p^n}[x]/(g)$ and $R=Z_{p^n}[a]$, where Z_{p^n} is the ring of integers modulo p^n , g is monic polynomial over Z_{p^n} and irreducible modulo p and a is an element of R of multiplicative order p^r-1 . In this case $\text{Aut } R$, the automorphism group of R , is cyclic and is of order r . These rings are uniquely determined by the triplet p, n, r ; they are called Galois rings and are denoted by $\text{GR}(p^n, r)$.

Let R be a finite completely primary ring. It is already known that any two coefficient subrings of R are conjugate (cf. [4]). Also if S is a coefficient subring of R ; then there exist $\pi_1, \dots, \pi_m$ in J and $\sigma_1, \dots, \sigma_m$ in $\text{Aut } S$ such that

$$R = S \oplus \sum_{i=1}^m \pi_i \quad (\text{as } S\text{-modules}) \quad \text{and} \quad \pi_i r = r^{\sigma_i} \pi_i$$

for all r in S and for all $i=1, \dots, m$. (This result is a direct consequence of theorems 2-2 and 2-4 in [6]). Moreover the automorphisms $\sigma_1, \dots, \sigma_m$ are uniquely determined by R and S (cf. [2]). Thus we call $\sigma_1, \dots, \sigma_m$ the associated automorphisms of R and the automorphism σ_i is called the automorphism associated with π_i . Throughout this paper, for a given finite completely primary ring R , we denote by T_R the set of all $(S, \pi_1, \dots, \pi_m)$ which come from the above description. In addition, let $F=R/J$, and let F^* and G_R denote the multiplicative group of units of F and R respectively.

2. THE CONSTRUCTION.

CONSTRUCTION A: Let S be a Galois ring of the form $GR(p^n, r)$ and F be S/pS . Also assume that s, t, w, m are non-negative integers such that $m=s+t+w$ and suppose that f is an injective function from $\{s+1, \dots, s+t\}$ to $\{s+1, \dots, m\}$. On the additive group $R=S \oplus P^n$, define the multiplication as follows:

$$(r_0, r_1, \dots, r_m)(s_0, s_1, \dots, s_m) = (r_0 s_0 + p^{n-1} \sum_{i=1}^s u_i r_i s_i^{\sigma_i} + p^{n-1} \sum_{i=s+1}^{s+t} r_i s_i^{\sigma_i}, r_0^* s_1 + r_1^* s_0^{\sigma_1}, \dots, r_0^* s_m + r_m^* s_0^{\sigma_m})$$

where u_i are elements of F , σ_i automorphisms of F such that $\sigma_i^2 = \text{id}_F$ for all $i=1, \dots, s$ and $\sigma_{f(i)} = \sigma_i^{-1}$ for all $i=s+1, \dots, s+t$ and r^* is the image of r under the canonical homomorphism from S to F .

It can be easily verified that R is a ring and it is commutative if and only if $\sigma_i = \text{id}_F$ for all $i=1, \dots, m$.

THEOREM 1: Let R be a finite completely primary ring. Then the product of any two zero divisors is an element of its coefficient subring S if and if it is one of the rings given by construction A.

PROOF: Let R be a finite completely primary ring with F contained in S and $(S, \pi_1', \dots, \pi_m')$ be an element of T_R . Since $S \cap S\pi_i' = 0$ and the product of any two zero divisors is in S , $p\pi_i' = 0$ for all $i=1, \dots, m$. But $\pi_i' \pi_j'$ is an element of pS ; thus $\pi_i' \pi_j'$ is an element of $p^{n-1}S$ for all $i, j=1, \dots, m$. Suppose $\pi_i' \pi_j', \pi_i' \pi_k'$ are non-zero elements of pS with $j \neq k$. Then $\pi_i' \pi_j' S = \pi_i' \pi_k' S = p^{n-1}S$ and we get $\pi_k' \pi_j' = \pi_i' \alpha$, where α is an element of $\langle a \rangle$. Thus $\pi_i' \pi_k' \alpha$ is an element of $\text{ann } \pi_i'$ and subsequently it is contained in

$$pS \oplus \sum_{h=1, h \neq j, k}^m S \pi_h'.$$

This implies that π_j' is an element of

$$pS \oplus \sum_{h=1, h \neq j}^m S \pi_h',$$

which contradicts the assumption that $(S, \pi_1', \dots, \pi_m')$ is an element of T_R . Therefore for all $i=1, \dots, m$, either $\pi_i' \pi_j'$ is zero for all $j=1, \dots, m$ or $\pi_i' \pi_j'$ is non-zero for only one $j=1, \dots, m$. Similarly, we prove that for all $i=1, \dots, m$, either $\pi_i' \pi_j'$ is zero for all $j=1, \dots, m$ or $\pi_i' \pi_j'$ is non-zero for exactly one j . Assume w is the number of π_i' such that $\pi_i' \pi_j'$ is zero for all $j=1, \dots, m$ and λ is the number of other π_i' . Let us reindex $\pi_1', \dots, \pi_m'$ in such a way that for each $i=1, \dots, \lambda$ there exists only one $j=1, \dots, m$ with $\pi_i' \pi_j' = p^{n-1} \alpha_{ij}$, where α_{ij} is an element of $\langle a \rangle$, and let f be the function from $\{1, \dots, \lambda\}$ to $\{1, \dots, m\}$ determined by $f(i)=j$. Clearly f is injective. Also, for all $i=1, \dots, m$

$$p^{n-1} a \alpha_{if(i)} = \pi_i' \pi_{f(i)}' a = a^{\sigma_i \sigma_{f(i)}} \pi_i' \pi_{f(i)}' = p^{n-1} a^{\sigma_i \sigma_{f(i)}} \alpha_{if(i)},$$

which implies that $\sigma_{f(i)} = \sigma_i^{-1}$ for all $i=1, \dots, \lambda$. Let s be the number of i in $\{1, \dots, \lambda\}$ such that $f(i)=i$ and t be $\lambda-s$. We reindex $\pi_1', \dots, \pi_\lambda'$ such that $f(i)=i$ for all $i=1, \dots, s$ and suppose $\alpha_{ii} = u_i$ for all $i=1, \dots, s$. Put $\pi_e = \pi_{f(i)}'$ for all $i=1, \dots, s$ and $\pi_e = \pi_{f(i)}' \alpha_e$ for all $i=s+1, \dots, m$, where if e is in the image of f , say $e=f(i)$, then

$$\alpha_e = \prod_{h=1}^j (\alpha_{f^{-1}(i) f^{-1}(i)}^{h-1})^{g(h)}, \text{ where } g(h) = (-1)^{j+h+1} \prod_{d=h}^{j-1} \sigma_{f^{-1}(i)}^d, \text{ and } \alpha_e = 1 \text{ otherwise.}$$

It is easy to see that $(S, \pi_1, \dots, \pi_m)$ is an element of T_R with $\pi_i \pi_{f(i)} = p^{n-1}$ for all $i=s+1, \dots, \lambda$. Now it follows that R is isomorphic to one of the rings given by construction A.

The converse is easy to check.

3. FINITE RINGS IN WHICH THE PRODUCT OF ANY TWO ZERO DIVISORS IS A POWER OF A FIXED PRIME.

LEMMA 1: Let R be a finite ring of characteristic p^n in which the product of any two zero divisors is a power of p . Then R is completely primary.

PROOF: Let x and y be zero divisors in R . To show that $x+y$ is a zero divisor, we can use the distributive properties to write $(x+y)^{2n}$ as a sum of products, each containing $2n$ factors (which are x 's or y 's). Since each xy or yx is of the form p^λ , each of the summands of $(x+y)^{2n}$ is product of the form $p^{\lambda_1}p^{\lambda_2}\dots p^{\lambda_n}=0$. Therefore $x+y$ is zero divisor and hence R is completely primary.

PROPOSITION 1: Let R be a finite ring of characteristic p^n in which the product of any two zero divisors is a power of p . Then R is completely primary with either $J^2=0$ or the coefficient subring of R is Z_{2^n} , where $n=2,3$.

PROOF: Suppose $J^2 \neq 0$; then there exist x, y in J with $xy = p^\lambda \neq 0$. Since for any unit α in R , αx is a zero divisor, we have $(\alpha x)y = p^\mu$. On the other hand, $xy = p^\lambda$ implies that $\alpha xy = \alpha p^\lambda$ and so $\alpha p^\lambda = p^\mu$. Without loss of generality, we can assume $\mu \geq \lambda$ and deduce that $p^\lambda(\alpha - p^{\mu-\lambda}) = 0$. Since $p^\lambda \neq 0$, we have $\alpha - p^{\mu-\lambda}$ is an element of J . If $\mu \neq \lambda$, this would imply that α is an element of J which is not possible; hence $\mu = \lambda$ and α is an element of $1+J$. However α is an arbitrary unit and therefore $G_R = 1+J$. Since $R = G_R \cup J$ (disjoint union), we have

$$|R| = |G_R| + |J| = |1+J| + |J| = 2|J|$$

Thus 2 divides $|R|$ and consequently $\text{char } R$ is 2^n . If $n \geq 4$, then 2, 6 are zero divisors of R with $(2)(6) = 12$ which is not a power of 2. Also $n=1$ implies that $J^2=0$. Thus $n=2,3$. Let $S = Z_{2^n}[a]$ be a coefficient subring of R , where a is an element of R of multiplicative order 2^t-1 and let x, y be elements of J with $xy = 2^\lambda \neq 0$. But $(ax)y = 2^\mu$ implies $a2^\lambda = 2^\mu$ and hence $a=1$. Thus the coefficient subring of R is Z_{2^n} with $n=2,3$.

4. THE ENUMERATION.

NOTATIONS: Retaining the above notations, assume k is the number of elements in $\{s+t+1, \dots, m\}$ which are not in the image of f . Let all the π_i in which i is not in the image of f be renamed as $\theta_1, \dots, \theta_k$ and assume $\tau_1, \dots, \tau_k$ are the respective automorphisms associated with them. Thus we suppose that $(S, \pi_1, \dots, \pi_{m-t}, \theta_1, \dots, \theta_k)$ is an element of T_R and $\sigma_1, \dots, \sigma_{m-t}, \tau_1, \dots, \tau_k$ are the automorphisms associated with $\pi_1, \dots, \pi_{m-t}, \theta_1, \dots, \theta_k$ respectively. We call (p, n, r, s, t, k, m, f) the invariants of R . In what follows we shall use these notations.

PROPOSITION 2: Let R be a finite completely primary ring in which the product of any two zero divisors is an element of its coefficient subring. Then $(S, \pi_1', \dots, \pi_{m-t}', \theta_1', \dots, \theta_k')$ is an element of T_R if and only if

$$\pi_i' = \lambda_i \pi_i + \sum_{\tau_j = \sigma_i} \xi_{ij} \theta_j + p^{n-1} \xi_i \quad (\text{after possible reindexing}),$$

$$\theta_i' = \sum_{\tau_j = \tau_i} \mu_j \theta_j + p^{n-1} \omega_i \quad (\text{after possible reindexing}).$$

where λ_i are elements of F^* and $\xi_{ij}, \xi_i, \mu_j, \omega_i$ are elements of F such that ξ_{ij} is zero if σ_j is not the trivial automorphism and ω_i is zero if τ_i is not the trivial automorphism.

PROOF: Using the fact that $\pi_i' a = a^{\sigma_i} \pi_i'$, we deduce that for all $i=1, \dots, m-k$, we have

$$\pi_i' = \sum_{\sigma_j = \sigma_i} \lambda_{ij} \pi_j' + \sum_{\tau_j = \sigma_i} \xi_{ij} \theta_j + p^{n-1} \xi_i.$$

where λ_{ij}, ξ_{ij} and ξ_i are elements of F such that ξ_i is zero if σ_i is not the trivial automorphism. For all $i=1, \dots, s+t$, $\text{lann } \pi_{(i)} = |J|/p^r$ and so $\pi_j' \pi_{(i)} = 0$ for all but one j , say $j=h$. Thus $\pi_h' \pi_{(i)}$ is a non-zero element of $p^{n-1}S$, $\pi_h' \pi_j = 0$ for all $j \neq f(i)$ and $\sigma_h = (\sigma_{(i)})^{-1} = \sigma_i$. Thus $\lambda_{ih} = 0$ for all j except $j=h$. Let us put $\lambda_{ih} = \lambda_i$ and redenote π_h' by π_i' . Therefore

$$\pi_i' = \lambda_i \pi_i' + \sum_{\tau_j = \sigma_i} \xi_{ij} \theta_j + p^{n-1} \xi_i.$$

We can prove the rest of the proposition by using a similar argument.

THEOREM 2: Let R, R' be finite completely primary rings constructed over the same coefficient subring S and having the same associated automorphisms. Suppose that $(J(R))^2$ and $(J(R'))^2$ are contained in S and R, R' have the same invariants p, n, r, s, t, k, m, f . Also suppose that $(S, \pi_1', \dots, \pi_{m-k}', \theta_1', \dots, \theta_k')$ is an element of T_R with $\pi_i'^2 = p^{n-1} v_i$ for all $i=1, \dots, s$. Then R is isomorphic to R' if and only if there exist isomorphisms ϕ_i from $S \oplus \pi_i$ to $S \oplus \pi_i'$ (after possible reindexing) for all $i=1, \dots, m-k$ such that $\phi_i(\pi_i) = \lambda_i \pi_i'$, where λ_i are elements of F such that

$$\lambda_i \lambda_i^{\sigma_i} = u_i^{p^j} v_i^{-1} \text{ and } \lambda_h \lambda_{f(h)}^{\sigma_h} = 1$$

for all $i=1, \dots, s$ and $h=s+1, \dots, s+t$, $0 \leq j < r$.

PROOF: Let ψ be an isomorphism from R to R' . Then $\psi(S)$ is a coefficient subring of R' and hence there exists a unit x in R' such that $\psi(S) = xSx^{-1}$. Let ϕ be the composition of the conjugation by x and ψ . Clearly ϕ is an isomorphism from R to R' which sends S to itself and thus $(S, \phi(\pi_1), \dots, \phi(\pi_{m-k}), \phi(\theta_1), \dots, \phi(\theta_k))$ is an element of T_R . Therefore for all $i=1, \dots, m-k$

$$\phi(\pi_i) = \lambda_i \pi_i' + \sum_{\tau_j = \sigma_i} \xi_{ij} \theta_j' + p^{n-1} \xi_i$$

where λ_i are elements of F and ξ_{ij}, ξ_i are elements of F such that ξ_i is zero if σ_i is not the trivial automorphism. For all $i=1, \dots, s$

$$p^{n-1} u_i^{p^j} = p^{n-1} \phi(u_i) = \phi(\pi_i^2) = (\phi(\pi_i))^2 = (\lambda_i \pi_i')^2 = p^{n-1} \lambda_i^{\sigma_i} \lambda_i.$$

Thus

$$\lambda_i \lambda_i^{\sigma_i} = u_i^{p^j} v_i^{-1} \text{ for some } 0 \leq j < r.$$

Also for all $i=s+1, \dots, s+t$

$$p^{n-1} = \phi(\pi_i^2) = (\phi(\pi_i))^2 = (\lambda_i \pi_i')^2 = \lambda_i \lambda_i^{\sigma_i} \pi_i'^2 = p^{n-1} \lambda_i \lambda_i^{\sigma_i}.$$

It is easy to see that, for all $i=1, \dots, s+t$, the mappings ϕ_i from $S \oplus S\pi_i$ to $S \oplus S\pi'_i$ determined by $\phi_i(\pi_i) = \lambda_i \pi'_i$ are isomorphisms.

Conversely, let ϕ_i be the isomorphisms from $S \oplus S\pi_i$ to $S \oplus S\pi'_i$ defined in the statement of the theorem, where $i=1, \dots, m-k$. It is easy to check that the mapping ϕ determined by

$$\phi(r_o + \sum_{i=1}^s r_i \pi_i + \sum_{i=1}^t \theta_i) = r_o + \sum_{i=1}^s \phi_i(\pi_i) + \sum_{i=1}^t \theta'_i$$

is an isomorphism from R to R' .

NOTATIONS: Let R be a finite completely primary ring in which the product of any two zero divisors is in its coefficient subring and let p, n, r, s, t, k, m, f be invariants of R . Assume ρ is the permutation on the maximal subset of $\{s+1, \dots, s+t\}$ which is stable under f and c is the number of cycles of ρ . Finally, let

$$\sigma_i = a^{p^{r_i}} \text{ for all } i = 1, \dots, s,$$

and N_i be the number of mutually non-isomorphic rings of the form $S \oplus S\pi_i$, with the same associated automorphisms σ_i , where $\pi_i^2 = p^{r_i} u_i$. Then from theorem 2 in [3], we have

$$N_i = \begin{cases} 1 & \text{if } p \text{ is even and } \sigma_i \text{ is the trivial automorphism,} \\ 2 & \text{if } p \text{ is odd and } \sigma_i \text{ is the trivial automorphism,} \\ p^{r/2} + 1 & \text{if } \sigma_i \text{ is not the trivial automorphism.} \end{cases}$$

THEOREM 3: The number of mutually non-isomorphic finite completely primary rings in which the product of any two zero divisors is in its coefficient subring, having the same invariants p, n, r, s, t, k, m, f and with the same associated automorphisms is

$$(p^r - 2)^{t-c} \prod_{i=1}^s N_i.$$

PROOF: If u_i, v_i are elements of F^* , define $u_i \sim v_i$ if and only if

$$u_i^{p^j} v_i^{-1} = \lambda_i^{p^{r_i} + 1}$$

for all $i=1, \dots, s$, where $0 \leq j < r$. By using similar method as in the proof of theorem 2 in [3], one can deduce that the number of equivalence classes of this equivalent relation is N_i . Define π_i, π'_i if and only if $\pi_i = \lambda_i \pi'_i$ for all $i=s+1, \dots, s+t$, where λ_i is an element of F^* such that $\lambda_i \lambda_{\rho(i)} = 1$. Let n_i be the number of the equivalence classes of this equivalent relation. Then $n_i=1$ if i is not in the image of f and $n_i=p^{t-2}$ if i is in the image of f . But when f restricted to $\{s+1, \dots, s+t\}$ the number of elements in the image of f is $t-c$. Thus

$$\prod_{i=s+1}^{s+t} n_i = (p^r - 2)^{t-c}.$$

In view of the last theorem the required number is

$$\left(\prod_{i=1}^s N_i \right) \left(\prod_{i=s+1}^{s+t} n_i \right) = (p^r - 2)^{t-c} \prod_{i=1}^s N_i.$$

COROLLARY: The finite ring of characteristic p^n in which the product of any two zero divisors is a power of p is completely determined by its associated automorphisms and its invariants.

REMARK: Let R be a finite ring which has a p -ring as its coefficient subring and the product of any two zero divisors of R is in its coefficient subring. By using similar argument as in the proof of lemma 1, one can prove that R is completely primary. Thus the construction and the enumeration of such rings is determined.

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