COMMON FIXED POINT THEOREMS FOR SEQUENCES OF FUZZY MAPPINGS

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(Received July 13, 1992 and in revised form February 15, 1993)

ABSTRACT. In this paper, we define g-contractive and g-contractive type fuzzy mappings and prove common fixed point theorems for sequences of fuzzy mappings on a complete metric linear space.

KEY WORDS AND PHRASES. Contractive-type fuzzy mapping, g-contractive fuzzy mapping, g-contractive type fuzzy mapping, fixed point, common fixed point.

1991 AMS SUBJECT CLASSIFICATION CODE. 54H25.

1. INTRODUCTION.

Fixed point theorems for fuzzy mappings were studied by Bose-Sahani, Butnariu, and others ([1]-[3]; [5]-[6]; [8]-[9]; [16]-[17]). While Weiss [17] studied a fixed point theorem for fuzzy sets, which is a fuzzy analogue of the Schauder-Tychonoff’s fixed point theorem, Heilpern [9] obtained a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorems for multi-valued mappings ([7], [10], [15]) and the well-known Banach fixed point theorem. A fixed point theorem for contractive type fuzzy mappings which is a generalization of the Heilpern’s result was given in [14]. In this paper, we define g-contractive and g-contractive type fuzzy mappings which are fuzzy analogues of g-contractive and g-contractive type mappings respectively ([11], [12]). For a mapping g of a complete metric linear space (X, d) into itself and a sequence (Fi)i∈[1,∞] of fuzzy mappings of X into W(X), we consider the following conditions (•) and (**);

• there exists a constant k with 0 ≤ k < 1 such that for each pair of fuzzy mappings F_i, F_j:X→W(X), D(F_i(x), F_j(y)) ≤ kd(g(x), g(y)) for all x, y ∈ X,

•• there exists a constant k with 0 ≤ k < 1 such that for each pair of fuzzy mappings F_i, F_j:X→W(X) and for any x ∈ X, {uz} ⊂ F_i(x) implies that there is {vy} ⊂ F_j(y) for all y ∈ X with D({uz}, {vy}) ≤ kd(g(x), g(y)).

We show that a sequence with the condition (•) satisfies the condition (••), that a sequence
with the condition \((***)\) has a common fixed point and consequently that a sequence with the condition \((*)\) has a common fixed point. These results are fuzzy analogues of common fixed point theorems for sequences of \(g\)-contractive and \(g\)-contractive type multi-valued mappings [11]. Consequently, we obtain as corollaries fixed point theorems for contractive fuzzy mappings [9] and contractive-type fuzzy mappings [14].

2. PRELIMINARIES.

We review briefly some definitions and terminologies needed ([4], [9], [16]). Let \((X,d)\) be a metric linear space (i.e., a complex or real vector space). A fuzzy set \(A\) in \(X\) is a function with domain \(X\) and values in \([0,1]\). (In particular, if \(A\) is an ordinary (crisp) subset of \(X\), its characteristic function \(\chi_A\) is a fuzzy set with domain \(X\) and values \([0,1]\)). Especially \([x]\) is a fuzzy set with membership function equal to a characteristic function of the set \([x]\). The \(\alpha\)-level set of \(A\), denoted by \(A_\alpha\), is defined by

\[A_\alpha = \{x : A(x) \geq \alpha\}\] where \(B\) denotes the closure of the (nonfuzzy) set \(B\). \(W(X)\) denotes the collection to all fuzzy sets \(A\) in \(X\) such that (i) \(A_\alpha\) is compact and convex in \(X\) for each \(\alpha \in [0,1]\) and (ii) \(\sup_{x \in X} A(x) = 1\). For \(A, B \in W(X)\), \(A \subseteq B\) means \(A(x) \leq B(x)\) for each \(x \in X\).

**DEFINITION 2.1.** Let \(A, B \in W(X)\). Then a metric \(D\) on \(W(X)\) is defined by

\[D(A, B) = \sup_{\alpha \in [0,1]} D(A_\alpha, B_\alpha)\] where \(H\) is the Hausdorff metric in the collection \(CP(X)\) of all nonempty compact subsets of \(X\).

**DEFINITION 2.2.** Let \(X\) be an arbitrary set and \(Y\) be any metric linear space. \(F\) is called a fuzzy mapping iff \(F\) is a mapping from the set \(X\) into \(W(Y)\).

A fuzzy mapping \(F\) is a fuzzy subset on \(X \times Y\) with a membership function \(F(x)(y)\). The function value \(F(x)(y)\) is the grade of membership of \(y\) in \(F(x)\). In case \(X = Y\), \(F(x)\) is a function from \(X\) into \([0,1]\). Especially for a multi-valued function \(f : X \to 2^X\), \(\chi f(x)\) is a function from \(X\) to \([0,1]\). Hence a fuzzy mapping \(F : X \to W(X)\) is another extension of a multi-valued function \(f : X \to 2^X\).

The concept of a fuzzy set provides a natural framework for generalizing many concepts of general topology to fuzzy topology.

**DEFINITION 2.3.** A family \(\mathcal{F}\) of fuzzy sets in a set \(X\) is called a fuzzy topology for \(X\) and the pair \((X, \mathcal{F})\) a fuzzy topological space, if (1) \(\chi_X \in \mathcal{F}\); (2) \(\chi_\emptyset \in \mathcal{F}\); (3) \(\cup A\in \mathcal{F}\) whenever each \(A\in \mathcal{F}\), \((A \in \mathcal{A})\); and (4) \(A \cap B \in \mathcal{F}\) whenever \(A, B \in \mathcal{F}\). The elements of \(\mathcal{F}\) are called open and their complements closed.

If a fuzzy set \(A\) in a (crisp) topological space \(X\) satisfies \(A(x) \geq \lim sup_{n \to \infty} A(x_n)\), where \((x_n)_{n \geq 1}\) is a sequence in \(X\) converging to a point \(x \in X\), then \(A\) is said to be closed [17]. The fact means that the closed fuzzy set \(A : X \to [0,1]\) is upper semicontinuous, i.e., a fuzzy set \(1 - A\) is lower semicontinuous [13]. Thus we are led to the following definition:

**DEFINITION 2.4 [17].** The induced fuzzy topology on a (crisp) topological space \((X, \mathcal{T})\), denoted by \(F(\mathcal{T})\), is the collection of all lower semicontinuous fuzzy sets in \(X\).

It is known that a fuzzy set \(A\) is open in a fuzzy topological space \((X, F(\mathcal{T}))\) [respectively, closed] if and only if for each \(\alpha \in [0,1], \{x \in X| A(x) > \alpha\}\) is open in a (crisp) topological space \((X, \mathcal{T})\) [respectively, \(\{x \in X| A(x) \geq \alpha\}\) is closed]. Recall that a function \(F(x) : X \to [0,1]\) is upper
semicontinuous for each $x \in X$, where $F$ is a fuzzy mapping defined on a metric linear space $(X, d)$ [14].

3. COMMON FIXED POINT THEOREMS FOR SEQUENCES OF FUZZY MAPPINGS.

In this section, we introduce the notions of $g$-contractive and $g$-contractive type fuzzy mappings. We show that a sequence of fuzzy mappings with the condition $(\ast)$ satisfies the condition $(\ast\ast)$, and a sequence with the condition $(\ast\ast)$ has a common fixed point. Consequently, we obtain that a $g$-contractive fuzzy mapping is $g$-contractive type, and that a $g$-contractive type fuzzy mapping has a fixed point.

**DEFINITION 3.1.** Let $g$ be a mapping from a metric linear space $(X, d)$ to itself. A fuzzy mapping $F: X \rightarrow W(X)$ is $g$-contractive if $D(F(x), F(y)) \leq kd(g(x), g(y))$ for all $x, y \in X$, for some fixed $k, 0 \leq k < 1$.

**PROPOSITION 3.2 [14].** Let $(X, d)$ be a complete metric linear space, $F: X \rightarrow W(X)$ a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

**DEFINITION 3.3 [14].** Let $(X, d)$ be a complete metric linear space. We call a fuzzy mapping $F: X \rightarrow W(X)$ contractive-type if for all $x, y \in X$, $\{u_x\} \subset F(x)$ there exists $\{v_y\} \subset F(y)$ for all $y \in X$ such that $D(\{u_x\}, \{v_y\}) \leq kd(x, y)$ for some fixed $k$, $0 \leq k < 1$.

A metric $D$ [respectively, Hausdorff metric $H$] is a metric on $W(X)$ [respectively, $CP(X)$] such that $D(\{x\}, \{y\}) = d(x, y)$ [respectively, $H(\{x\}, \{y\}) = d(x, y)$]. Hence $D$ [respectively, $H$] is a generalization of the metric $d$ to fuzzy sets [respectively, crisp sets].

**DEFINITION 3.4.** Let $g$ be a mapping from a complete metric linear space $(X, d)$ to itself. We call a fuzzy mapping $F: X \rightarrow W(X)$ $g$-contractive type if for all $x \in X$, $\{u_x\} \subset F(x)$ there exists $\{v_y\} \subset F(y)$ for all $y \in X$ such that $D(\{u_x\}, \{v_y\}) \leq kd(g(x), g(y))$ for some fixed $k$, $0 \leq k < 1$.

We consider an example of a $g$-contractive type fuzzy mapping which is not contractive-type.

**EXAMPLE 3.5.** Let $(X, d)$ be a Euclidean metric space $[0, \infty)$, $d(|\cdot|)$. Define $F: X \rightarrow W(X)$ as follows:

$$F(x)(z) = \begin{cases} 1, & 0 \leq z \leq 2x \\ 0, & z > 2x \end{cases}$$

and define $g: [0, \infty) \rightarrow [0, \infty)$ by $g(z) = 3z$. Then $F$ is not contractive-type but $g$-contractive type.

**THEOREM 3.6.** Let $g$ be a mapping from a complete metric linear space $(X, d)$ to itself. If $(F_i)_{i=1}^{\infty}$ is a sequence of fuzzy mappings of $X$ into $W(X)$ satisfying the condition $(\ast)$, then $(F_i)_{i=1}^{\infty}$ satisfies the condition $(\ast\ast)$.

**PROOF.** Let $x, y \in X$. If $D(F_i(x), F_j(y)) \leq kd(g(x), g(y))$ for some fixed $k$, $0 \leq k < 1$, then $H(F_i(x), F_j(y)) \leq kd(g(x), g(y))$ for each $\alpha \in [0, 1]$. Define $(F_i)_{\alpha}: X \rightarrow CP(X)$ by $(F_i)_{\alpha}(x) = F_i(x)_{\alpha}$ for each $\alpha \in [0, 1]$. Then $H((F_i)_{\alpha}(x), (F_j)_{\alpha}(y)) = H(F_i(x)_{\alpha}, F_j(y)_{\alpha}) \leq kd(g(x), g(y))$ for each $\alpha \in [0, 1]$. Thus, for each $x \in X, u_x \in (F_i)_{\alpha}(x)$, there exists $v_y \in (F_j)_{\alpha}(y)$ for all $y \in X$ such that $H(\{u_x\}, \{v_y\}) \leq kd(g(x), g(y))$ for each $\alpha \in [0, 1]$. Since $u_x \in F_i(x)_{\alpha}$ and $v_y \in F_j(y)_{\alpha}$, there exists $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that $D(\{u_x\}, \{v_y\}) \leq kd(g(x), g(y))$ for some fixed $k$, $0 \leq k < 1$.

The converse of Theorem 3.6 does not hold in general.

**EXAMPLE 3.7.** Let $g$ be an identity mapping from a Euclidean metric space $[0, \infty)$, $d(|\cdot|)$ to itself. Let $(F_i)_{i=1}^{\infty}$ be a sequence of fuzzy mappings from $[0, \infty)$ into $W([0, \infty))$, where $F_i(x): [0, \infty) \rightarrow [0, 1]$ is defined as follows:
If \( x = 0 \), \( F_i(x)(z) = \begin{cases} 1, & 0 \leq z \leq \frac{\varepsilon}{2} \\ 0, & z \notin 0 \end{cases} 
\)
otherwise,
\[
F_i(x)(z) = \begin{cases} 1, & 0 \leq z \leq \frac{\varepsilon}{2} \\ \frac{1}{2}, & \frac{\varepsilon}{2} \leq z \leq ix \\ 0, & z > ix. \end{cases}
\]

Then the sequence \((F_i)_{i=1}^\infty\) satisfies the condition (**), but does not satisfy the condition (*).

**COROLLARY 3.8** [14]. Let \((X,d)\) be a complete metric linear space. If \(F:X \rightarrow W(X)\) is a contractive fuzzy mapping, then it is contractive-type.

**COROLLARY 3.9.** Let \(g\) be a mapping from a complete metric linear space \((X,d)\) to itself. If \(F:X \rightarrow W(X)\) is a \(g\)-contractive fuzzy mapping, then \(F\) is \(g\)-contractive type.

Weiss [17] proved a generalization to fuzzy sets of the Schauder-Tychonoff theorem by means of the classical Schauder-Tychonoff theorem, and Butnariu [2] proved that a convex and closed fuzzy mapping \(F\) defined over a nonempty convex compact subset of a real topological vector space, locally convex and Hausdorff separated, has a fixed point. Also he showed that a \(F\)-continuous fuzzy mapping defined over a nonempty convex compact subset of a \(n\)-dimensional Euclidean space \(R^n(n \in N)\) has a fixed point.

Now we prove our main theorem which extends the result of Heilpern [9] on fuzzy contraction mappings and the result of Lee-Cho [14] on contractive-type fuzzy mappings to the case of a sequence of fuzzy mappings on a complete metric linear space.

**THEOREM 3.10.** Let \(g\) be a non-expansive mapping from a complete metric linear space \((X,d)\) to itself. If \((F_i)_{i=1}^\infty\) is a sequence of fuzzy mappings of \(X\) into \(W(X)\) satisfying the condition (**), then there exists \(p \in X\) such that \(\{p\} \subset F_i(P)\).

**PROOF.** Let \(x_0 \in X\). Then we can choose \(x_1 \in X\) with \(d(x_0,x_1) > 0\) such that \(\{x_1\} \subset F_1(x_0)\) by Proposition 3.2. By the condition (**), there exists \(x_2 \in X\) such that \(\{x_2\} \subset F_2(x_1)\) with
\[
D(\{x_1\}, \{x_2\}) \leq kd(g(x_0),g(x_1)) \leq kd(x_0,x_1),
\]
for some fixed \(k, 0 < k < 1\). Inductively, we obtain a sequence \((x_n)_{n=1}^\infty\) in \(X\) such that \(\{x_{n+1}\} \subset F_{n+1}(x_n)\) and
\[
D(\{x_1\}, \{x_{n+1}\}) \leq kd(g(x_{n-1}),g(x_n))
\]
for all \(n\). This leads to \(\{x_{n+1}\} \subset F_{n+1}(x_n)\) and \(d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)\) for all \(n\). Since
\[
d(x_n, x_{n+1}) = D(\{x_n\}, \{x_{n+1}\}) \leq \frac{k^{n+1}}{1-k} D(\{x_0\}, \{x_1\}) \leq \frac{k^n}{1-k} d(x_0, x_1)
\]
for all \(n > m\) and \(d(x_n, x_m) \rightarrow 0\) as \(m,n \rightarrow \infty\). By the completeness of \(X\) we find an element \(p \in X\) with \(x_n \rightarrow p\) as \(n \rightarrow \infty\). Let \(F_m\) be an arbitrary member of \((F_i)_{i=1}^\infty\). Since \(\{x_n\} \subset F_n(x_{n-1})\) for all \(n\), there exists \(\{v_n\} \subset F_m(p)\) such that
\[
D(\{x_n\}, \{v_n\}) \leq kd(g(x_{n-1}), g(p)) \leq kd(x_{n-1}, p).
\]
But we have \(d(p,v_n) \leq d(p,x_n) + d(x_n,v_n) \leq d(p,x_n) + kd(x_{n-1}, p)\) which implies \(d(p,v_n) \rightarrow 0\) as \(n \rightarrow \infty\). Since \(F_m(p) : X \rightarrow [0,1]\) is upper semicontinuous, \(\limsup_{n \rightarrow \infty} F_m(p)(v_n) \leq F_m(p)(p)\). Since \(F_m(p)(v_n) = 1\) for all \(n\), \(F_m(p)(p) = 1\). Hence \(\{p\} \subset F_m(p)\) for all \(m\), that is, \(\{p\} \subset \bigcap_{i=1}^\infty F_i(p)\).

**REMARK.** The sequence \((F_i)_{i=1}^\infty\) in Example 3.7 has a common fixed point \(z = 0\).

**COROLLARY 3.11.** Let \(g\) be a non-expansive mapping from a complete metric linear space \((X,d)\) to itself. If \((F_i)_{i=1}^\infty\) is a sequence of fuzzy mappings of \(X\) into \(W(X)\) satisfying the condition (*), then there exists \(p \in X\) such that \(\{p\} \subset \bigcap_{i=1}^\infty F_i(p)\).

**COROLLARY 3.12.** Let \(g\) be a non-expansive mapping from a complete metric linear space \((X,d)\) to itself. If \(F:X \rightarrow W(X)\) is a \(g\)-contractive type fuzzy mapping, then there exists \(p \in X\) such that \(\{p\} \subset F(p)\).

**COROLLARY 3.13** [14]. Let \((X,d)\) be a complete metric linear space. If \(F:X \rightarrow W(X)\) is a contractive-type fuzzy mapping, then there exists \(p \in X\) such that \(\{p\} \subset F(p)\).
COROLLARY 3.14 [9]. Let $X$ be a complete metric linear space and $F$ a fuzzy mapping from $X$ to $W(X)$ satisfying the following condition; there exists $q \in (0,1)$ such that $D(F(x), F(y)) \leq qd(x, y)$ for each $x, y \in X$. Then there exists $p \in X$ such that $\{p\} \subseteq F(p)$.

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