ABSTRACT. It is shown that there exist a \( \sigma \)-weakly closed operator algebra \( \hat{A} \) generated by finite rank operators and a \( \sigma \)-weakly closed operator algebra \( \hat{B} \) generated by compact operators such that the Fubini product \( \hat{A} \hat{\otimes} \hat{B} \) contains properly \( \hat{A} \hat{\otimes} \hat{B} \).

KEY WORDS AND PHRASES. The slice map problem.

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1. INTRODUCTION.

In [6] Kraus initiated the slice map problem for \( \sigma \)-weakly closed operator spaces. By an operator space we mean a norm closed linear subspace of \( L(H) \), the operators of a Hilbert space \( H \). As stated in the introduction of [9], the slice map problem is of interest because a number of questions concerning tensor products of \( \sigma \)-weakly closed operator spaces are special cases of the slice map problem [4–9].

A \( \sigma \)-weakly closed operator space \( A \) is said to have Property \( S_\sigma \) if \( A \hat{\otimes} F B = A \hat{\otimes} B \) for any \( \sigma \)-weakly closed subspace \( B \) [6]. Kraus [9] first gave \( \sigma \)-weakly closed operator spaces not having Property \( S_\sigma \). Effros et al. [3] also characterized \( \sigma \)-weakly closed operator spaces having Property \( S_\sigma \). One of useful theorems [7, Theorem 2.1] for the slice map problem says that a \( \sigma \)-weakly closed unital operator algebra generated by finite rank operators has Property \( S_\sigma \) (cf. [10]). In this paper, we show that the condition "unital" is essential in the theorem.

2. MAIN RESULT.

For operator spaces \( A \) and \( B \), let \( A \hat{\otimes} B \) denote the norm closed linear span of \( \{a \hat{\otimes} b : a \in A \text{ and } b \in B\} \). If \( A \) and \( B \) are \( \sigma \)-weakly closed, let \( A \hat{\otimes} B \) denote the \( \sigma \)-weakly closed linear span of \( \{a \hat{\otimes} b : a \in A \text{ and } b \in B\} \).

Let \( X \) and \( Y \) be von Neumann algebras. For \( g \in X_* \), the predual of \( X \), the right slice map \( R_g \) associated with \( g \) is a unique bounded linear map from \( X \hat{\otimes} Y \) to \( Y \) such that \( R_g(z \hat{\otimes} y) = < z, g > y \). For \( h \in Y_* \), the left slice map \( L_h \) from \( X \hat{\otimes} Y \) to \( X \) is a unique bounded linear map such that \( L_h(z \hat{\otimes} y) = < y, h > z \). Let \( A \) and \( B \) be \( \sigma \)-weakly closed linear subspaces of \( X \) and...
Y, respectively. We define the Fubini product $A \otimes_F B$ of A and B by $A \otimes_F B = \{ z \in X \otimes Y : R_g(z) \in B, L_h(z) \in A \}$ for every $g \in X$, $h \in Y$. The space $A \otimes_F B$ does not depend on $X \otimes Y$ [6, Remark 1.2].

Let $A$ be a $C^*$-algebra. If we assume that $A$ acts universally on a Hilbert space $H$, the second dual $A^{**}$ of $A$ can be identified with the $\sigma$-weak closure $B$ of $A$ in $L(H)$. In this case, the weak* topology on $A^{**}$ coincides with the $\sigma$-weak topology on $B$.

The following example shows that the condition "containing the identity" is necessary in Theorem 2.1 of [7].

EXAMPLE. There exist a $\sigma$-weakly closed operator algebra $\tilde{A}$ generated by finite rank operators on a Hilbert space $H$ and a $\sigma$-weakly closed operator algebra $\tilde{B}$ generated by compact operators on $H$ such that $\tilde{A} \subset \tilde{B}$.

PROOF. Let $c_0$ denote the $C^*$-algebra of all complex sequences that converge to zero. Davie [1] constructed a closed linear subspace $A_0$ of $c_0$ satisfying the following properties: (1) $A_0$ does not have the approximation property in the sense of Grothendieck; (2) $A_0$ contains a dense linear subspace $A_1$ with the norm topology such that each element has finite support, where each element of $c_0$ is identified with a function whose domain is the set of all positive integers.

Since $c_0^{**}$ is *-isomorphic to $\ell^\infty$, the von Neumann algebra of all bounded sequences, we assume that $c_0^{**}$ acts on the Hilbert space $\ell^2$ in the usual way. Let $A$ denote the $\sigma$-weak closure of $A_0$ in $c_0^{**}$. For a closed linear subspace $D_0$ of $c_0$, let $D$ denote the $\sigma$-weak closure of $D_0$ in $c_0^{**}$. We note that $(c_0 \hat{\otimes} c_0)^{**} = c_0^{**} \hat{\otimes} c_0^{**}$ and $A \hat{\otimes} D \subset c_0^{**} \hat{\otimes} c_0^{**}$. Put $F(A_0, D_0, c_0 \hat{\otimes} c_0) = \{ z \in c_0 \hat{\otimes} c_0 : R_g(z) \in D_0, L_h(z) \in A_0 \}$ for every $g \in c_0, h \in c_0$.

By the same argument in the proof of [9, Theorem 5.8] (with a $C^*$-algebra $A$ replaced by an operator space $A$), we can choose a closed linear subspace $B_0$ of $c_0$ such that $F(A_0, B_0, c_0 \hat{\otimes} c_0)$ contains properly $A_0 \hat{\otimes} B_0$. Let $B$ be the $\sigma$-weak closure of $B_0$ in $c_0^{**}$. Since $A \cap c_0 = A_0$ and $B \cap c_0 = B_0$, we have $F(A_0, B_0, c_0 \hat{\otimes} c_0) \subset (A \hat{\otimes} B) \cap (c_0 \hat{\otimes} c_0)$. The opposite inclusion is trivial. It follows that $F(A_0, B_0, c_0 \hat{\otimes} c_0) = (A \hat{\otimes} B) \cap (c_0 \hat{\otimes} c_0)$. Since $A \hat{\otimes} B$ is identified with the weak* closure of $A_0 \hat{\otimes} B_0$ in $(c_0 \hat{\otimes} c_0)^{**}$, we have $(A \hat{\otimes} B) \cap (c_0 \hat{\otimes} c_0) = A \hat{\otimes} B_0$. Hence $A \hat{\otimes} F B$ contains properly $A \hat{\otimes} \tilde{B}$.

Let $H = \ell^2 \oplus \ell^2$. Put $\widehat{A} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in A \right\}$ and $\tilde{B} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in B \right\}$. Since $A_1$ consists of finite rank operators on $\ell^2$, it is easy to see that $\tilde{A}$ is a $\sigma$-weakly closed operator algebra generated by finite rank operators on $H$. Since $c_0$ consists of compact operators on $\ell^2$, $\tilde{B}$ is a $\sigma$-weakly closed operator algebra generated by compact operators on $H$. Then

$$\tilde{A} \hat{\otimes} F \tilde{B} \simeq \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : a \in A \hat{\otimes} F B \right\}$$

and

$$\tilde{A} \hat{\otimes} \tilde{B} \simeq \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : a \in A \hat{\otimes} \tilde{B} \right\}.$$ 

Hence $\tilde{A} \hat{\otimes} F \tilde{B}$ contains properly $\tilde{A} \hat{\otimes} \tilde{B}$. This completes the proof.

Let $K$ be the $C^*$-algebra of all compact operators on a separable infinite dimensional Hilbert space. An operator space $A$ is said to have the operator approximation property if
there exists a net \( \{ \phi_\alpha \} \) of finite rank linear maps from \( A \) to itself such that \( \phi_\alpha \otimes id_K(z) \rightarrow z \) in norm for every \( z \in A \otimes_K K \) [2].

Using techniques in the proof of Example, we restate Theorem 5.5 of [9] in a slightly different form.

**PROPOSITION.** Let \( A_0 \) be a closed linear subspace of a \( C^* \)-algebra \( D \) and let \( A \) be the weak* closure of \( A_0 \) in \( D^{**} \).

Then the following statements are equivalent:

1. \( A_0 \) has the operator approximation property;
2. \( A_0 \otimes B_0 = (A \otimes_F B) \cap (D \otimes K) \) for any closed linear subspace \( B_0 \) of \( K \).

**PROOF.** We may assume that \( D \) and \( K \) act in their universal representations. We note that \( D \otimes_K K \subseteq D^{**} \otimes_K K^{**} = (D \otimes K)^{**} \). Let \( B_0 \) be a closed linear subspace of \( K \) and let \( B \) be the weak* closure of \( B_0 \) in \( K^{**} \). Put \( F(A_0, B_0, D \otimes K) = \{ z \in D \otimes K: R_g(z) \in B_0, L_h(z) \in A_0 \) for every \( g \in D^*, h \in K^* \} \). Since \( A \cap D = A_0 \) and \( B \cap K = B_0 \), we have \( F(A_0, B_0, D \otimes K) \supseteq (A \otimes_F B) \cap (D \otimes K) \). The opposite inclusion is trivial. It follows that \( F(A_0, B_0, D \otimes K) = (A \otimes_F B) \cap (D \otimes K) \). Then (2) holds if and only if \( F(A_0, B_0, D \otimes K) = A_0 \otimes B_0 \) for any closed subspace \( B_0 \) of \( K \). Hence the same argument in the proof of [9, Theorem 5.5] (with a \( C^* \)-algebra \( A \) replaced by an operator space \( A \)) implies that (1) and (2) are equivalent.

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**REFERENCES**

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