THE MEET OF LOCALLY CONNECTED GROUP TOPOLOGIES

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ABSTRACT. The collection of all group topologies on a given group is a complete lattice partially ordered by inclusion. The purpose of this paper is to develop a convenient method of defining the meet of an arbitrary collection of group topologies and demonstrate that the meet of a collection of locally connected group topologies is also locally connected.

KEY WORDS AND PHRASES. Topological groups, quotient spaces, locally connected, translatable and semicontinuous topologies.

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1. INTRODUCTION.

There has been considerable interest, by Graev [1], Comfort [2] and others, in finding group topologies on a given group. An approach is to start with a topology on a given group and derive an associated group topology. In [3] Roelcke and Dierolf developed a method for finding a neighborhood basis for the finest group topology contained in the intersection of an arbitrary collection of topologies on a group. In general, however, this basis does not consist of open sets. In [4] Clark and Schneider studied the meet in the lattice of group topologies and developed a convenient method for finding a basis of open sets for the meet of a countable collection of group topologies on an abelian group.

In this paper we will develop a convenient method for finding an open neighborhood basis for the finest group topology contained in any topology on a group. This method will enable us to find a neighborhood basis for the meet of an arbitrary collection of group topologies and demonstrate that the meet of an arbitrary collection of locally connected group topologies is also locally connected.

2. PRELIMINARY RESULTS.

Let G be a group. Then it is well known that $\mathcal{F}$ is a fundamental system for G iff $\mathcal{F}$ is a collection of subsets of G each containing the identity and satisfying the following properties:

- 1) If $U, V \in \mathcal{F}$ then there exists $W \in \mathcal{F}$ such that $W \subset U \cap V$.
- 2) If $U \in \mathcal{F}$ and $a \in U$ then there exists $V \in \mathcal{F}$ such that $aV \subset U$.
- 3) If $U \in \mathcal{F}$ and $x \in G$ then there exists $V \in \mathcal{F}$ such that $xVx^{-1} \subset U$.
- 4) If $U \in \mathcal{F}$ then there exists $V \in \mathcal{F}$ such that $V^{-1} \subset U$.
- 5) If $U \in \mathcal{F}$ then there exists $V \in \mathcal{F}$ such that $VV \subset U$.

If $\mathcal{F}$ is a fundamental system then there exists a unique topology t on G such that t is a group topology having $\mathcal{F}$ as a basis at the identity.

DEFINITION 1. Let G be a group and $\mathcal{T}$ be a collection of subsets of G each containing the identity of G . Then $\mathcal{T}$ is a translatable system for G iff $\mathcal{T}$ satisfies properties 1, 2 and 3 above and by [5] $\mathcal{T}$ is a semifundamental system for G iff $\mathcal{T}$ satisfies properties 1 through 4 above.

In [5] Clay defined a semicontinuous topology and showed that a basis of open

neighborhoods at the identity is a semifundamental system. She also showed that if $\mathcal{V}$ is a semifundamental system on a group G then there is a unique topology t on G such that t is a semicontinuous topology having $\mathcal{V}$ as a basis at the identity.

DEFINITION 2. A translatable group is a group G endowed with a topology t such that Ua and aU are elements of t for all $U \in t$ and $a \in G$. If (G, t) is a translatable group then we say that t is a translatable topology on G .

It is easy to see that if t is a translatable topology then a basis of open neighborhoods at the identity is a translatable system and if $\mathcal{T}$ is a translatable system for a group G then there is a unique topology t on G such that t is a translatable topology having $\mathcal{T}$ as a basis at the identity.

DEFINITION 3. Let G be a group and t any topology on G . The Graev operator assigns to the pair (G, t) the pair $(G, g(t))$ where $g(t)$ is the finest group topology on G contained in t .

The collection of all group topologies on a given group is a complete lattice partially ordered by inclusion. The join of an arbitrary collection of group topologies $\{t_\alpha\}_{\alpha \in \Delta}$ has $\{V : V \in t_\alpha \text{ for some } \alpha \in \Delta\}$ as a subbasis and the meet is the join of $\{t : t \subset t_\alpha \text{ for all } \alpha \in \Delta \text{ and } t \text{ is a group topology}\}$. Samuel pointed out in [6] that $\bigcap_{\alpha \in \Delta} t_\alpha$ may not be a group topology. Therefore $\bigwedge_{\alpha \in \Delta} t_\alpha \neq \bigcap_{\alpha \in \Delta} t_\alpha$ (in general) but $g(\bigcap_{\alpha \in \Delta} t_\alpha) \subset t_\alpha$ for all $\alpha \in \Delta$ and $g(\bigcap_{\alpha \in \Delta} t_\alpha)$ is a group topology. Therefore $g(\bigcap_{\alpha \in \Delta} t_\alpha) = \bigwedge_{\alpha \in \Delta} t_\alpha$.

If $G_i = G$ for all $i \in \mathbb{N}$ then we let $\bigoplus_{i=1}^\infty G$ denote $\bigoplus_{i=1}^\infty G_i$, $\prod_{i=1}^\infty G$ denote $\prod_{i=1}^\infty G_i$ and define $m_\infty: \bigoplus_{i=1}^\infty G \rightarrow G$ by $m_\infty(\langle a_i \rangle) = a_1 a_2 a_3 \cdots a_n$ where $a_m = e$ for all $m > n$. Let T be a translatable topology on $\bigoplus_{i=1}^\infty G$, t be a translatable topology on G , and S be the topology inherited by $\bigoplus_{i=1}^\infty G$ as a subspace of $\prod_{i=1}^\infty G$ when $\prod_{i=1}^\infty G$ is endowed with the box topology with t on each factor. Then $q_\omega(T)$ shall denote the quotient topology on G induced by $m_\infty: (\bigoplus_{i=1}^\infty G, T) \rightarrow G$ and $q_\omega[t]$ shall denote the quotient topology $q_\omega(S)$.

THEOREM 1. If T is a translatable topology on $\bigoplus_{i=1}^\infty G$ then the quotient map $m_\infty: (\bigoplus_{i=1}^\infty G, T) \rightarrow (G, q_\omega(T))$ is open.

PROOF: Let $V \in T$ and $\langle a_i \rangle \in m_\infty^{-1}(m_\infty(V))$ then there exists $\langle b_i \rangle \in V$ such that $m_\infty(\langle b_i \rangle) = m_\infty(\langle a_i \rangle)$. Since $\langle a_i \rangle$ and $\langle b_i \rangle$ are in $\bigoplus_{i=1}^\infty G$ there exists m such that $a_n = e$ and $b_n = e$ for all $n > m$ and $b_1 b_2 b_3 \cdots b_m = a_1 a_2 a_3 \cdots a_m$.

Let $h_1 = b_1^{-1} a_1$ and $h_i = b_i b_{i+1} b_{i+2} \cdots b_m a_m^{-1} a_{m-1}^{-1} \cdots a_1^{-1}$ for all $i=3, 4, 5, \dots, m$. Let $u = \langle e, h_1^{-1}, h_3^{-1}, h_4^{-1}, \dots, h_{m-1}^{-1}, h_m^{-1}, e, e, e, \dots \rangle$ and $w = \langle h_1, h_3, h_4, \dots, h_m, e, e, e, \dots \rangle$ then $u, w \in \bigoplus_{i=1}^\infty G$ and $\langle a_i \rangle = u \langle b_i \rangle w \in uVw \in T$ since T is translatable.

Let $\langle x_i \rangle \in uVw$. Then there exists $\langle y_i \rangle \in V$ such that $\langle x_i \rangle = u \langle y_i \rangle w$ and there exists n_y such that $y_i = e$ for all $i > n_y$. Let $k = \max \{m, n_y\}$ then:

$$\begin{aligned} m_\infty(\langle x_i \rangle) &= x_1 x_2 x_3 \cdots x_k = (y_1 h_1)(h_1^{-1} y_2 h_3)(h_3^{-1} y_3 h_4) \cdots (h_{m-1}^{-1} y_{m-1} h_m)(h_m^{-1} y_m) y_{m+1} y_{m+2} \cdots y_k \\ &= y_1 y_2 y_3 \cdots y_k \in m_\infty(V) \quad \blacksquare \end{aligned} \quad (2.1)$$

COROLLARY 1. Let $n \in \mathbb{N}$ and define $m_n: \prod_{i=1}^n G \rightarrow G$ by $m_n(a_1, a_2, a_3, \dots, a_n) = a_1 a_2 a_3 \cdots a_n$. Let T be a translatable topology on $\prod_{i=1}^n G$ then the quotient map $m_n: (\prod_{i=1}^n G, T) \rightarrow G$ is open.

3. THE GRAEV OPERATOR.

Let $n \in \{0, 1, 2, \dots\}$, P_n be the set of all permutations on $\{n+1, n+2, n+3, \dots\}$ and σ_i denote $\sigma^{-1}(i)$ for all $i \in \mathbb{N}$ and $\sigma \in P_n$. Suppose s is a semicontinuous topology on G and $\mathcal{S}$ is a semifundamental system for s . Let $\langle A_i \rangle$ be a sequence of elements of $\mathcal{S}$ and:

$$P(\langle A_i \rangle, n) = \bigcup_{\substack{\sigma \in P_n \\ j \in \mathbb{N}}} (A_1 \times A_2 \times A_3 \times \dots \times A_n \times A_{\sigma_{n+1}} \times A_{\sigma_{n+2}} \times \dots \times A_{\sigma_{n+j}} \times e \times e \dots) \quad (3.1)$$

Then $\{P(\langle A_i \rangle, n) : A_i \in \mathcal{S} \text{ and } n \in \{0, 1, 2, \dots\}\}$ is a translatable system for a topology, C_s , on $\bigoplus_{i=1}^{\infty} G$. (Introduced by Bradd Clark, C_s is known as the cross topology due to the "shape" of the basic neighborhoods of the identity element.) Now if we let $C(\langle A_i \rangle) = \bigcup_{\sigma \in P_n} [\bigcup_{n \in \mathbb{N}} (A_{\sigma_1} A_{\sigma_2} A_{\sigma_3} \dots A_{\sigma_n})]$ then $\mathcal{C}(s) = \{C(\langle A_i \rangle) : A_i \in \mathcal{S}\}$ is a neighborhood basis of the identity for $q_{\infty}(C_s)$. This follows from the fact that the quotient map $m_{\infty} : (\bigoplus_{i=1}^{\infty} G, C_s) \longrightarrow G$ is open.

THEOREM 2. $q_{\infty}(C_s) = g(s)$.

PROOF: One can easily see that $\mathcal{C}(s)$ satisfies properties 1, 3 and 4 of a fundamental system. Therefore let us consider properties 2 and 5. Let $U = C(\langle U_i \rangle)$ be an element of $\mathcal{C}(s)$ and $a \in U$ then:

(2) There exists $n \in \mathbb{N}$ and $\delta \in P_0$ such that $a \in U_{\delta_1} U_{\delta_2} U_{\delta_3} \dots U_{\delta_n}$. Therefore we can find $a_i \in U_{\delta_i}$ for $i = 1, 2, 3, \dots, n$ such that $a = a_1 a_2 a_3 \dots a_n$ and we can find $V_i \in \mathcal{S}$ such that $V_i a_i \subset U_{\delta_i}$ for all $i = 1, 2, 3, \dots, n$. Let $b_i = a_1 a_{i+1} a_{i+2} \dots a_n$ then there exists $U'_i \in \mathcal{S}$ and $W \in \mathcal{S}$ such that $b_i U'_i \subset V_i b_i$ for all $i = 1, 2, 3, \dots, n$ and $W \subset \bigcap_{i=1}^n U'_i$. Now $U_{\delta_1} U_{\delta_2} U_{\delta_3} \dots U_{\delta_n} \supset a W^n$ and we can find, for all $i \in \mathbb{N}$, $W_i \in \mathcal{S}$ such that $W_i \subset U_{\delta_{n+1}} \cap W^n$. Therefore $a[C(\langle W_i \rangle)] \subset U$.

(5) There exists $V_i \in \mathcal{S}$ for $i = 1, 3, 5, \dots$ such that $V_i \subset U_i \cap U_{i+1}$. Let:

$$\langle A_i \rangle = \langle U_1, U_3, U_5, \dots \rangle, \quad \langle B_i \rangle = \langle U_2, U_4, U_6, \dots \rangle \text{ and } V = C(\langle V_i \rangle). \quad (3.2)$$

Then $V^2 \subset [C(\langle A_i \rangle)][C(\langle B_i \rangle)] \subset U$.

Therefore $q_{\infty}(C_s)$ is a group topology and since s is translatable $q_{\infty}(C_s) \subset s$. Let $U \in g(s)$ such that $e \in U$ and suppose $\mathcal{S}$ is the set of all open neighborhoods of e . Then there exists $V_1 \in g(s)$ such that $e \in V_1$ and $(V_1)^3 \subset U$. We can also find $V_i \in g(s)$ such that $e \in V_i$ and $(V_i)^3 \subset V_{i-1}$ for all $i = 2, 3, 4, \dots$.

Let $\sigma \in P_0$ and $k_{m,n} = \min\{\sigma_m, \sigma_{m+1}, \sigma_{m+2}, \dots, \sigma_n\}$ (σ_i denotes $\sigma^{-1}(i)$ for all $i \in \mathbb{N}$). Then $V_{\sigma_1} V_{\sigma_2} \subset (V_{k_{1,2}})^2 \subset (V_{k_{1,2}})^3 \subset U$, $V_{\sigma_1} V_{\sigma_2} V_{\sigma_3} \subset (V_{k_{1,3}})^3 \subset U$ and $V_{\sigma_1} V_{\sigma_2} V_{\sigma_3} V_{\sigma_4} \subset (V_{k_{1,2}})^2 (V_{k_{3,4}})^2 \subset (V_{k_{1,4}})^3 \subset U$.

Suppose then that $V_{\sigma_1} V_{\sigma_2} V_{\sigma_3} \dots V_{\sigma_i} \subset (V_{k_{1,i}})^3 \subset U$ for all $i \leq n-1$ and $\sigma \in P_0$.

Let $\sigma_j = k_{1,n}$ then:

$$V_{\sigma_1} V_{\sigma_2} V_{\sigma_3} \cdots V_{\sigma_n} = V_{\sigma_1} V_{\sigma_2} V_{\sigma_3} \cdots V_{\sigma_{j-1}} V_{\sigma_j} V_{\sigma_{j+1}} \cdots V_{\sigma_n} \quad (3.3)$$

$$< (V_{k_{1,j-1}})^3 V_{k_{1,n}} (V_{k_{j+1,n}})^3 > < (V_{k_{1,n}})^3 > \subset U. \quad (3.4)$$

Therefore $C(\langle V_1 \rangle) \subset U$ and $g(s) \subset q_\omega(C_s) \subset s$. Therefore $g(s) = q_\omega(C_s)$ since $q_\omega(C_s)$ is a group topology ■

If G is abelian then $\mathcal{C}(s) = \{ \bigcup_{n \in \mathbb{N}} (V_1 V_2 V_3 \cdots V_n) \mid V_i \in \mathcal{V} \}$ and by a proof similar to the one above we can prove the following corollary.

COROLLARY 2 If G is abelian then $g(s) = q_\omega[s]$.

Suppose then that we have an arbitrary collection of group topologies $\{t_\alpha\}_{\alpha \in \Delta}$ on a given group. Since $\bigcap_{\alpha \in \Delta} t_\alpha$ is semicontinuous we have that $\mathcal{C}(\bigcap_{\alpha \in \Delta} t_\alpha)$ is a fundamental system for $\bigwedge_{\alpha \in \Delta} t_\alpha$.

In [7] Clark and Schneider defined $S(t)$ to be the finest semicontinuous topology on G contained in t and gave a concrete description of $S(t)$. Since $g(t)$ is semicontinuous $g(S(t)) = g(t)$. Therefore we can also use Theorem 2 to prove the following corollary.

COROLLARY 3. If t is any topology on a group G and $C_{S(t)}$ is Clark's Cross topology with respect to $S(t)$ then $q_\omega(C_{S(t)}) = g(t)$.

4. LOCAL CONNECTEDNESS.

Suppose A and B are connected subsets of a translatable group G then aB is connected for all $a \in A$. Let $x \in B$ then $Ax \subset AB$ and Ax is connected. Therefore $AB = Ax \cup AB = Ax \cup [\bigcup_{a \in A} (aB)]$ is connected since $ax \in Ax \cap aB$ for all $a \in A$. Therefore we have the following theorem.

THEOREM 3. Suppose t is a locally connected and translatable topology on G then $q_\omega[t]$ is also locally connected.

In a similar fashion we can prove the following corollary.

COROLLARY 4. If t is a locally connected and translatable topology on G and C_t is Clark's Cross topology with respect to t then $q_\omega(C_t)$ is also locally connected.

Suppose then that $\{t_\alpha\}_{\alpha \in \Delta}$ is a collection of locally connected group topologies on G then $\bigcap_{\alpha \in \Delta} t_\alpha$ is also locally connected and therefore $g(\bigcap_{\alpha \in \Delta} t_\alpha) = \bigwedge_{\alpha \in \Delta} t_\alpha$ is also locally connected.

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