ABSTRACT. Let \( M_n \) be the classes of regular functions \( f(z) = z^{-1} + a_0 + a_1 z + \cdots \) defined in the annulus \( 0 < |z| < 1 \) and satisfying \( \Re \frac{f^{n+1}(z)}{f^n(z)} > 0 \), \( n \in \mathbb{N}_0 \), where \( f^0(z) = f(z) \), \( f^1(z) = \frac{f(z) - f(z_0)}{z - z_0} \), \( f^n(z) = \frac{f(f^{n-1}(z))}{f'(z)} \), and \( * \) is the Hadamard convolution. We denote by \( \Gamma_n = M_n \cup \Gamma \), where \( \Gamma \) denotes the class of functions of the form \( f(z) = z^{-1} + \sum_{k=1}^{\infty} |a_k| z^k \). We obtained that relates the modulus of the coefficients to starlikeness for the classes \( M_n \) and \( \Gamma_n \), and coefficient inequalities for the classes \( \Gamma_n \).

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1. INTRODUCTION

Let \( \Sigma \) denote the class of function of the form \( f(z) = z^{-1} + a_0 + a_1 + \ldots \) that are regular in \( 0 < |z| < 1 \) with a simple pole at \( z = 0 \). In [1] Dernek defined the classes \( M_n \) of functions \( f \in \Sigma \) and satisfying the condition

\[
\Re \frac{f^{n+1}(z)}{f^n(z)} > 0 \quad (|z| < 1, n \in \mathbb{N}_0)
\]

where \( f^0(z) = f(z) \), \( f^1(z) = (z^{-1} - z(z - 1)^{-2}) f(z) = -zf'(z) \) and \( f^n(z) = \frac{f(f^{n-1}(z))}{f'(z)} = z^{-1} + (-1)^n \sum_{k=1}^{\infty} k^n a_k z^k \). \( M_0 \) and \( M_1 \) are known classes of univalent functions that are meromorphically starlike and convex respectively. He proved that \( M_{n+1} \subset M_n \) for each \( n \in \mathbb{N}_0 \). Since \( M_0 = \Sigma^* \), the element of \( M_n \) are univalent and starlike. Further \( \Gamma_n = M_n \cap \Gamma \), where \( \Gamma \) denotes the subclass of \( \Sigma \) consisting of functions of the form

\[
f(z) = z^{-1} - \sum_{k=1}^{\infty} |a_k| z^k.
\]

In section 2 coefficient inequalities are obtained for the classes \( M_n \) and \( \Gamma_n \), similar problems were treated in [2] and [4].

2. COEFFICIENT INEQUALITIES

We begin with a theorem that relates the modulus of the coefficients to starlikeness. Our results are generalizations of the results obtained by Pommerenke in [3].
THEOREM 1. Let \( f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \). If \( \sum_{k=1}^{\infty} k^{n+1}|a_k| < 1 \), then \( f \in M_n, (n \in \mathbb{N}_0) \).

PROOF. We define \( w(z) \) in \( 0 < |z| < 1 \) by

\[
\frac{F^{n+1} f(z)}{F^n f(z)} = \frac{1 - w(z)}{1 + w(z)}.
\]  

(2.1)

It suffices to show that \( |w(z)| < 1 \). We have from (2.1)

\[
|w(z)| = \left| \frac{F^n f(z) - F^{n+1} f(z)}{F^n f(z) + F^{n+1} f(z)} \right|
\]

\[
= \left| \frac{(-1)^n \sum_{k=1}^{\infty} (k+1) k^n a_k z^{k+1}}{2 - (-1)^n \sum_{k=1}^{\infty} (k-1) k^n a_k z^{k+1}} \right|
\]

\[
\leq \frac{\sum_{k=1}^{\infty} (k+1) k^n |a_k|}{2 - \sum_{k=1}^{\infty} (k-1) k^n |a_k|}.
\]

The last expression is bounded by 1 if

\[
\sum_{k=1}^{\infty} (k+1) k^n |a_k| < 2 - \sum_{k=1}^{\infty} (k-1) k^n |a_k|
\]

which reduces to

\[
\sum_{k=1}^{\infty} k^{n+1} |a_{k+1}| \leq 1.
\]  

(2.2)

But (2.2) is true by hypothesis. Hence \( |w(z)| < 1 \) and the theorem is proved.

Special cases of Theorem 1 have been proved by Pommerenke [3, p. 274]:

COROLLARY 1: If we substitute \( n = 0 \) in the above theorem, then we have \( f \in \sum \) and \( \sum_{k=1}^{\infty} k |a_k| \leq 1 \), therefore \( f \) is starlike univalent in \( 0 < |z| < 1 \).

COROLLARY 2: If we substitute \( n = 1 \) in the above theorem, then we have \( f \in \sum \) and \( \sum_{k=1}^{\infty} k^2 |a_k| \leq 1 \), therefore \( f \) is convex univalent in \( 0 < |z| < 1 \).

THEOREM 2: A function \( f(z) = \frac{1}{z} - \sum_{k=1}^{\infty} |a_k| z^k \) is in \( \Gamma_n \) if and only if

\[
\sum_{k=1}^{\infty} k^{n+1} |a_k| < 1, \quad (n \in \mathbb{N}_0).
\]

PROOF: In view of Theorem 1, it suffices to show that the only if part. Assume that \( f \in \Gamma_n \). Let \( z \) be complex numbers. If \( \Re(z) > 0 \) then \( \Re(1/z) > 0 \). Thus from (1.1) we obtain

\[
0 < \Re \left\{ \frac{F^n f(z)}{F^{n+1} f(z)} \right\} \leq \left| \frac{F^n f(z)}{F^{n+1} f(z)} \right|
\]

\[
= \left| \frac{1 - (-1)^n \sum_{k=1}^{\infty} k^n |a_k| z^{k+1}}{1 - (-1)^{n+1} \sum_{k=1}^{\infty} k^{n+1} |a_k| z^{k+1}} \right|
\]

\[
\leq \frac{1 + \sum_{k=1}^{\infty} k^n |a_k|}{1 - \sum_{k=1}^{\infty} k^{n+1} |a_k|}.
\]

Hence \( \sum_{k=1}^{\infty} k^{n+1} |a_k| \leq 1 \) and the proof is complete.

This result is thus generalization of the result obtained by Pommerenke [3, p. 275].

COROLLARY 3: If \( f \in \Gamma_n \), then \( |a_k| \leq \frac{1}{k^{n+1}}, (n \in \mathbb{N}_0) \), with equality for

\( f_k(z) = \frac{1}{z} - \frac{1}{k^{n+1}} z^k, (n \in \mathbb{N}_0) \).
REFERENCES


