ON THE SPEED OF CONVERGENCE OF ITERATION OF A FUNCTION

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ABSTRACT. Let $f_n(x)$ be the $n^{th}$ iterate of a function in some interval $[0, c]$. It is known that if $f(x) \sim x - x^a$, $a > 1$, then $f_n(x) \sim A n^a$ for some $A$ and $a$. In this paper we prove a converse of this theorem: The rate of convergence of the iterates determines the form of a function.

KEY WORDS AND PHRASES: Iterations of a function, slow convergence.


Let $f(x)$ be a real valued function; denote the $n^{th}$ iterate of $f(x)$ by $f_n(x)$, i.e., $f_0(x) = x$, $f_{n+1}(x) = f(f_n(x))$. If on some interval $[0, c]$ the function $f$ is continuous and satisfies the inequality $0 < f(x) < x$ for $x > 0$, then $\lim_{n \to \infty} f_n(x) = 0$ for every $x \in [0, c]$. Indeed, for every such $x$, $f_n(x)$ is monotonically decreasing and it is easy to see that the limit must be 0. The rates of convergence of the sequence $f_n(x)$ have been studied extensively, see Ostrowski [1] or Seneta [3]. If $f'(0) < 1$, the sequence converges at least geometrically fast: There is a constant $0 < \gamma < 1$ such that $f_n(x) < \gamma^n$ for large $n$. The situation is more delicate when $f'(0) = 1$. This is known as "slow convergence problem". A. M. Ostrowski [1] has proved the following result:

THEOREM 1. Suppose $f(x)$ is a continuous increasing function on some interval $[0, c]$ such that $0 < f(x) < x$ for $0 < x \leq c$. If $f(x) = x - K x^p + o(x^p)$ as $x \to 0$, where $K > 0, p > 1$, then for all $x \in [0, c]$

$$\lim_{n \to \infty} n^a f_n(x) = A$$

where $a - p + 1 = 0$ and $a - K A^{p-1} = 0$.

These sufficient conditions for $f_n(x)$ to behave like $A n^{-a}$ are also, in some sense, necessary, as the next theorem shows. We recall that a function $f(x)$ is said to be concave if

$$f(ux + (1-u)y) \geq uf(x) + (1-u)f(y)$$

for any $x, y$ in the domain and $0 \leq u \leq 1$. If the function $f(x)$ is concave, then "the slopes decrease": For $x_1 < x_2 < x_3$ we have
\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \frac{f(x_3) - f(x_2)}{x_3 - x_2}
\]

For the proof see, for instance, Rudin [2].

**THEOREM 2.** Suppose \( f(x) \) is an increasing, continuous, and concave function on \([0, c]\) satisfying \( 0 < f(x) < x \) \((x \neq 0)\). Suppose also that for some fixed \( a > 0, \lim_{n \to \infty} n^a f_n(x) \) exists and is different from 0 for every \( x \in [0, c] \). Then for every \( \epsilon > 0 \) there is \( c_\epsilon > 0 \) such that

\[
x - x^p - \epsilon \leq f(x) \leq x - x^p + \epsilon, \text{ for } 0 < x \leq c_\epsilon,
\]

where \( a - ap + 1 = 0 \), i.e., \( p = 1 + 1/a \).

We need the following lemma.

**LEMMA.** Let \( 0 < V < v \) and let \( g(x) = sx - L \) be a linear function with slope \( 0 < s < 1 \) such that \( g(v) = V \). Let \( w \) be the number such that \( g(w) = w \) and let \( w < z < v \). Put

\[
N = N(v, V, s, z) = \frac{\log \left( \frac{z + \frac{v - V}{1-s} - v}{1-s} \right) - \log \left( \frac{v - V}{1-s} \right)}{\log(s)}.
\]

If \( k > N \) then \( g_k(v) < z \), and if \( k > N \) then \( g_k(v) > z \).

**PROOF.** Put \( t_1 = v - V = v - g(V) \) and \( t_k + 1 = g_k(v) - g_{k+1}(v), k = 1, 2, \ldots \). Then

\[
t_{k+1} = s \left[ g_{k-1}(v) - g_k(V) \right] = st_k
\]

and so

\[
t_1 + t_2 + \ldots + t_k = t_1 \left( 1 + s + s^2 + \ldots + s^k - 1 \right) = t_1 \frac{1 - s^k}{1 - s} = \frac{v - V}{1-s} \left( 1 - s^k \right).
\]

But then

\[
g_k(v) = v - (t_1 + t_2 + \ldots + t_k) = v - \frac{v - V}{1-s} \left( 1 - s^k \right) = \frac{v - V}{1-s} s^k + v - \frac{v - V}{1-s}.
\]

This is a decreasing sequence in \( k \), hence \( g_k(v) < z \) is equivalent to \( k > N \), and \( g_k(v) > z \) is equivalent to \( k < N \).

**PROOF of Theorem 2.** It is enough to show that, under the hypothesis of the theorem,

\[
\lim_{x \to 0^+} \frac{\log(x - f(x))}{\log(x)} = p = 1 + 1/a
\]

(1)

We break the proof into two parts: I. \( \lim \inf \geq p \), and II. \( \lim \sup \leq p \).

Proof of I. Let \( t = \lim \inf \) and assume that \( t < p \). We notice, by the way, that \( t \geq 1 \) because \( |\log(x - f(x))| > |\log(x)| \) for \( 0 < x < 1 \), and \( p > 1 \). Thus there exists a sequence \( c > x_1 \geq x_2 \geq \ldots \to 0 \) such that
Since the function $f(x)$ is concave and $f(0) = 0$, the ratio $f(x)/x$ is a decreasing function of $x$. From (2) it follows that $f(x_k) = x_k - x_k^t$, and thus

$$s_k = \frac{f(x_k)}{x_k} = 1 - x_k^t$$

increases as $k \to \infty$, hence $|\log s_k|$ decreases as $k \to \infty$. We may thus require that the sequence \{x_k\} satisfies

$$\frac{\log x_{k+1}}{\log s_{k+1}} \geq 2 \frac{\log x_k}{\log s_k} \quad \text{and} \quad \frac{\log x_k}{\log s_k} \geq k$$

Let $k$ be fixed. The slope of the line joining $(0, f(0))$ and $(x_k, f(x_k))$ is equal to $s_k = 1 - x_k^t$. If $x > x_k$, then the slope of the line joining $(x_k, f(x_k))$ and $(x, f(x))$ is less than $s_k$ (the function $f(x)$ is concave), hence $f(x) \leq s_k x$ for $x \geq x_k$. Define function $g(x)$ by

$$g(x) = s_k x \quad \text{if} \quad x_k \leq x \leq x_{k-1}, \quad k = 2, 3, \ldots$$

We have just proved that $f(x) \leq g(x)$, so $f_m(x) \leq g_m(x)$ for all integers $m$ ($f$ is monotone, i.e., $f_2(x) \leq f(g(x))$, \ldots), $f_m(x) = f(f_{m-1}(x)) \leq f(g_{m-1}(x)) \leq g_m(x)$). Let $n_k$ be the smallest integer such that $g_{n_k}(x_{k-1}) \leq x_k$. We apply the Lemma with $\nu = x_{k-1}$, $V = f(x_k) = x_{k-1} - x_{k-1}^t$, $z = x_k$, $s = V/\nu$. A simple calculation leads to

$$n_k \leq \frac{\log x_k - \log x_{k-1}}{\log s_k} + 1.$$

We remark that if $y < x_k$ then $g_{n_k}(y) \leq x_k$. Indeed, if $y < x_k$, the result is immediate since $g(x) < x$; if $x_k \leq y < x_{k-1}$, then $g_n(y) < x_k$ for some $n \leq n_k$, so $g_{n_k-n}(g_n(y)) \leq x_k$. Thus

$$f_{n_2}(x_1) \leq g_{n_2}(x) \leq x_2$$
$$f_{n_2+n_3}(x_1) \leq g_{n_3}(f_{n_2}(x_1)) \leq x_3$$
$$\cdots \cdots \cdots$$
$$f_{n_2+n_3+\ldots+n_k}(x_1) \leq g_{n_k}(f_{n_2+\ldots+n_{k-1}}(x_1)) \leq x_k.$$

Setting $N_k = n_2 + n_3 + \ldots + n_k$, the last inequality in (5) becomes $f_{N_k}(x_1) \leq x_k$, which implies that for any $b > 0$

$$N_k^b f_{N_k}(x_1) \leq N_k^b x_k.$$

By hypothesis of the theorem, if $b > a$ then the left side of (6) goes to $\infty$ as $k \to \infty$. To obtain the desired contradiction we will show that the right hand side of (6) goes to 0 as $k \to \infty$ for some $b > a$. We now estimate $N_k$. From (4) we obtain

$$N_k = \sum_{m=2}^k n_m = k + \sum_{m=2}^k \frac{\log x_m - \log x_{m-1}}{\log s_m} \leq k + \sum_{m=2}^k \frac{\log x_m}{\log s_m}$$
However, the requirement (3) gives

\[ \frac{\log x_k - 1}{\log s_k - 1} \leq \frac{1}{2} \frac{\log x_k}{\log s_k} \leq \left( \frac{1}{2} \right)^2 \left( \frac{\log x_k}{\log s_k} \right) \]

\[ \log \frac{x_{k-2}}{s_{k-2}} \leq \left( \frac{1}{2} \right)^2 \log \frac{x_{k-1}}{s_{k-1}} \leq \left( \frac{1}{2} \right)^2 \frac{\log x_k}{\log s_k} \]

\[ \log \frac{x_2}{s_2} \leq \left( \frac{1}{2} \right)^2 \log \frac{x_3}{s_3} \leq \ldots \leq \left( \frac{1}{2} \right)^{k-2} \frac{\log x_k}{\log s_k} \]

Substituting these in (7) we obtain

\[ N_k \leq k + \sum_{m=2}^{k} \left( \frac{1}{2} \right)^{m-1} \frac{\log x_k}{\log s_k} \leq \frac{\log x_k}{\log s_k} + 2 \frac{\log x_k}{\log s_k} = 3 \frac{\log x_k}{\log s_k} \]

the last inequality being justified by (3). It is thus sufficient to show that for some \( b > a \)

\[ \lim_{k \to \infty} \left( \frac{\log x_k}{\log s_k} \right)^b x_k = 0, \]

or, what comes to the same thing

\[ \lim_{k \to \infty} \frac{x_k^{1/b} \log x_k - 1}{\log s_k - 1} = 0. (8) \]

Now, \( t_k - 1 \) is monotonically decreasing to \( t - 1 \) (see (2)), hence \( t_k - 1 < t - 1 + \epsilon \) for arbitrary \( \epsilon \) and \( k \) sufficiently large. Thus

\[ 1 - x_k^{t_k - 1} \leq 1 - x_k^{t_k - 1 + \epsilon} \]

or

\[ |\log s_k| = |\log (1 - x_k^{t_k - 1})| \geq |\log (1 - x_k^{t_k - 1 + \epsilon})|, \]

so

\[ \frac{x_k^{1/b} \log x_k}{\log s_k} \leq \frac{x_k^{1/b} \log x_k}{\log (1 - x_k^{t_k - 1 + \epsilon})} \]

for arbitrary \( \epsilon \) and \( k \) sufficiently large. To establish (8) it is sufficient now to show that there exists \( \epsilon > 0 \) and \( b > a \) so that

\[ \lim_{x \to 0+} \frac{x^{1/b} \log x}{\log (1 - x^{t-1+\epsilon})} = 0, \]

where \( t < 1 + 1/a \). Since \( \log(1 + u) \sim u \) as \( u \to 0 \), the expression in (9) is less than

\[ -2x^{1/b + 1 - t} \log(x) \]

(10)
for sufficiently small $x$. But $t < 1 + (1/a^2)$ hence there is $\epsilon > 0$ such that $1 - t + (1/a^2) - \epsilon > 0$, and so for some $b > a$ the exponent in (10) is strictly positive, i.e., (9) holds ($x^r \log x \to 0$ for any $r > 0$). This proves I.

Proof of II. Again, we argue by contradiction. Assume that there exists a sequence $c > x_1 \geq x_2 \geq \cdots \to 0$ such that

$$\frac{\log(x_k - f(x_k))}{\log(x_k)} = t_k \to t > p. \quad (11)$$

Without loss of generality we may assume that

$$\left| \log \left( \frac{x_k^{t+1}}{x_k} \right) - \log \left( \frac{x_k^{t+1}}{x_k} \right) \right| \geq \frac{1}{2} \left| \log \left( \frac{x_k^{t+1}}{x_k} \right) \right| \quad (12)$$

$$\left| \log \left( \frac{1}{2} + \frac{1}{2} x_k^{t+1} x_k^{-t_k} \right) \right| \geq \left| \log \frac{3}{4} \right| \quad (13)$$

$$\left( x_k - x_k + 1 \right) \left( \frac{x_k^{t+1}}{x_k} - x_k^{t+1} \right) \geq \frac{1}{2} x_k^{t+1} \quad (14)$$

It follows from (11) that $f(x_k) = x_k - x_k^{t_k}$. Let $h(x)$ be the function defined by

$$h(x) = \begin{cases} f(x_k) & \text{if } x = x_k \\ \text{linear} & \text{if } x_k + 1 \leq x \leq x_k \\ 0 & \text{if } x = 0 \end{cases}$$

Since the function $f(x)$ is concave, we see that $f(x) \geq h(x)$ and so, as in the proof of part I, $f_m(x) \geq h_m(x)$ for all integers $m$. Define two integers $n_k$ and $m_k$ as follows: $n_k$ is the largest integer such that $h_n(x_k) \geq x_k + 1$ and $m_k$ is the largest integer such that $h_m(x_k) \geq \frac{1}{2} \left( x_k + x_k + 1 \right)$. We now obtain estimates on $m_k$ using the Lemma. In this case $v = x_k$, $V = x_k^{t_k}$, $z = \frac{1}{2} \left( x_k + x_k + 1 \right)$, and

$$s = \frac{f(x_k) - f(x_k + 1)}{x_k - x_k + 1} = 1 - \frac{x_k^{t_k} - x_k^{t+k+1}}{x_k - x_k^{t_k} + 1} = s_k$$

Applying the Lemma, we obtain

$$m_k \geq \log \left[ \frac{1/2 (x_k + x_k + 1) + x_k^{t_k} x_k - x_k^{t+k+1}}{x_k - x_k^{t_k} + 1} - x_k \right] - \log \left( \frac{x_k^{t_k} x_k - x_k^{t+k+1}}{x_k - x_k^{t_k} + 1} \right) - 1$$

After direct simplification this reduces to

$$m_k \geq \left( \log s_k \right)^{-1} \log \left( \frac{1}{2} + \frac{1}{2} x_k^{t+k+1} x_k^{-t_k} \right) - 1.$$

We apply (13) and the fact that $s_k \to 1$ to obtain

$$m_k \geq c_1 \frac{1}{1 - s_k} = c_1 \frac{x_k - x_k + 1}{x_k^{t_k} - x_k^{t+k+1}}.$$
for large $k$, where $c_1$ is a constant. Finally, from (14) we obtain

$$m_k \geq c_2 x_k \left(1 - t_k\right)^{1 - t_k}$$

(15)

for sufficiently large $k$, where $c_2$ is some constant. For $k \geq 2$ set $N_k = n_1 + n_2 + \ldots + n_{k-1} + m_k$. It follows from the definition of $n$'s and $m$'s that

$$h(x) \geq \frac{1}{2} (x_k + x_{k+1}) \geq \frac{1}{2} x_k$$

hence for $b > 0$ we have

$$N_k f_{N_k}(x_1) \geq N_k h_{N_k}(x_1) \geq \frac{1}{2} N_k x_k$$

(16)

Since $naf_n(x_1)$ converges to a limit that is different from 0, it follows that as $k \to \infty$, $N_k f_{N_k}(x_1) \to 0$ if $b < a$. But $N_k > m_k$, so (16) implies that $m_k x_k \to 0$ as $k \to \infty$ whenever $b < a$. From (15) we see that

$$m_k x_k \geq c_2 x_k^{b(1 - t_k) + 1}$$

(17)

Now, $1 + 1/a < t_k$ so $a(1 - t_k) + 1 < 0$, and thus $b(1 - t_k) + 1 < 0$ for some $b < a$ and $k$ sufficiently large. We see from (17) that for such $b$, $m_k x_k \to 0$. This contradiction completes the proof.

A word or two regarding the concavity assumption in the Theorem 2. The assumption is certainly needed in the proof. The result is also not true without it. The idea is this: Construct an arbitrary sequence $0 < x_n < 1$, $x_n \downarrow 0$ at an arbitrary rate. It is easy to see that one can construct a function $f(x)$ such that $f(1) = x_1$ and $f(x_n) = x_{n+1}$ (Just draw a picture). The values of $f(x)$ at other point can be taken completely arbitrarily so that the conclusion of Theorem 2 need not hold.

REFERENCES


