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ON THE EXISTENCE AND UNIQUENESS PROBLEMS OF SOLUTIONS FOR SET-VALUED AND SINGLE-VALUED NONLINEAR OPERATOR EQUATIONS IN PROBABILISTIC NORMED SPACES

SHIH-SEN CHANG

Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
People's Republic of China

YEOL JE CHO

Department of Mathematics
Gyeongsang National University
Jinju 660-701, Korea

FAN WANG

Jiangsu Industrial and
Commercial School
Nantong, Jiangsu 226004
People's Republic of China

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ABSTRACT. In this paper, we introduce the concept of more general probabilistic contractors in probabilistic normed spaces and show the existence and uniqueness of solutions for set-valued and single-valued nonlinear operator equations in Menger probabilistic normed spaces.

KEY WORDS AND PHRASES. Menger PM-spaces, t-norms, probabilistic contractors, set-valued and single-valued operator equations, fixed points.

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1. INTRODUCTION.

In [1], M. Altman showed the existence and uniqueness of solutions for set-valued and single-valued nonlinear operator equations in Banach spaces by means of the concepts of contractors and contractor directions. Since Altman, Lee and Padgett ([5]-[7]) introduced the concept of random contractors with random nonlinear majorant functions and showed the existence and uniqueness of solutions for random operator equations by random contractors.

In this paper, we introduce the concept of more general probabilistic contractors in probabilistic normed spaces and show the existence and uniqueness of solutions for set-valued and single-valued nonlinear operator equations in Menger probabilistic normed spaces. Our results extend and improve the corresponding results of Altman [1], Chang ([3], [4]), Lee and Padgett [6].

2. PRELIMINARIES.

Throughout this paper, let $\mathbf{R} = (-\infty, +\infty)$ and $\mathbf{R}^+ = [0, +\infty)$. A mapping $\mathcal{F}: \mathbf{R} \to \mathbf{R}^+$ is called a distribution function if it is nondecreasing and left-continuous with $\inf \mathcal{F} = 0$ and $\sup \mathcal{F} = 1$. We also denote $\mathfrak{D}$ and H by the set of all distribution functions and the specific distribution function defined by

$$H(t) = \begin{cases} 1, t > 0, \\ 0, t \leq 0, \end{cases}$$

respectively.

A function $\Delta: [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

$(\Delta - 1)$ $\Delta(a, 1) = a$ and $\Delta(0, 0) = 0$;

$(\Delta - 2)$ $\Delta(a, b) \leq \Delta(c, d)$ for $a \leq c$ and $b \leq d$;

$(\Delta-3)$ $\Delta(a,b)=\Delta(b,a)$;

$(\Delta-4)$ $\Delta(\Delta(a,b),c)=\Delta(a,\Delta(b,c))$.

A triplet $(X,\mathcal{F},\Delta)$ is called a Menger probabilistic normed space (briefly, a Menger PN-space) if X is a real vector space, $\mathcal{F}$ is a mapping from X into $\mathcal{D}$ (for $x\in X$, the distribution function $\mathcal{F}(x)$ is denoted by F_x and $F_x(t)$ is the value of F_x at $t\in\mathbf{R}$) and Δ is a t-norm satisfying the following conditions:

$(PN-1)$ $F_x(0)=0$;

$(PN-2)$ $F_x(t)=H(t)$ for all $t\geq 0$ if and only if $x=0$;

$(PN-3)$ $F_{\alpha x}(t)=F_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha\in\mathbf{R},\alpha\neq 0$;

$(PN-4)$ $F_{x+y}(t_1+t_2)\geq(\Delta F_x(t_1),F_y(t_2))$ for all $x,y\in X$ and $t_1,t_2\in\mathbf{R}^+$.

A non-Archimedean Menger probabilistic normed space (briefly, a N.A. Menger PN-space) is a triplet $(X,\mathcal{F},\Delta)$, where $(X,\mathcal{F},\Delta)$ is a Menger PN-space and the t-norm Δ satisfies the following condition instead of $(PN-4)$:

$(PN-5)$ $F_{x+y}(max\{t_1,t_2\})\geq\Delta(F_x(t_1),F_y(t_2))$ for all $x,y\in X$ and $t_1,t_2\in\mathbf{R}^+$.

Note that if $(X,\mathcal{F},\Delta)$ is a Menger PN-space with the t-norm Δ satisfying the following condition:

$$\sup_{0<t<1}\Delta(t,t)=1, \tag{2.1}$$

then $(X,\mathcal{F},\Delta)$ is a real metrizable Hausdorff vector topological space with the topology τ induced by the family of neighborhoods,

$$\{U_y(\varepsilon,\lambda):y\in X,\varepsilon>0,\lambda>0\}, \tag{2.2}$$

where

$$U_y(\varepsilon,\lambda)=\{x\in X:F_{x-y}(\varepsilon)>1-\lambda\}.$$

Let $(X,\mathcal{F},\Delta)$ be a Menger PN-space with the t-norm Δ satisfying the condition (2.1) and Ω_X be a family of all nonempty probabilistically bounded τ-closed subsets of X. For any given $A,B\in\Omega_X$, define the distribution functions $F_{A,B}$ and F_A by

$$F_{A,B}(t)=\sup_{s<t}\ \Delta(\inf_{a\in A}\ \sup_{b\in B}\ F_{a-b}(s),\ \inf_{b\in B}\ \sup_{a\in A}\ F_{a-b}(s))$$

and

$$F_A(t)=\sup_{s<t}\ \sup_{a\in A}\ F_a(s)$$

for all $s,t\in\mathbf{R}$, respectively.

Then, from the definitions of $F_{A,B}(t)$ and $F_A(t)$, we have the following:

LEMMA 2.1. Let $(X,\mathcal{F},\Delta)$ be a Menger PN-space (resp., an N.A. Menger PN-space) with the t-norm Δ satisfying the condition (2.1) and $A\in\Omega_X$. Then we have the following:

(1) $F_A(0)=0$;

(2) $F_A(t)=1$ for all $t>0$ if and only if $\theta\in A$;

(3) $F_{\lambda A}(t)=F_A\left(\frac{t}{|\lambda|}\right)$ for all $\lambda\in\mathbf{R},\lambda\neq 0$;

(4) For any $A,B\in\Omega_X$ and $\theta\in B, F_A(t)\geq F_{A,B}(t)$ for all $t\in\mathbf{R}$;

(5) If the t-norm Δ is continuous, then we have

$F_{A+x}(t_1+t_2)\geq\Delta(F_x(t_1),F_A(t_2))$ (resp., $F_{A+x}(max\{t_1,t_2\})\geq\Delta(F_x(t_1),\ F_A(t_2))$) for all $t_1,t_2\in\mathbf{R}^+$ and $x\in X$.

Recall that a sequence $\{x_n\}$ in X converges to a point $x\in X$ in the topology τ (denoted by $x_n\xrightarrow{\tau}x$) if

$$\lim_{n\to\infty}F_{x_n-x}(t)=H(t)\text{ for all }t\geq 0.$$

A sequence $\{x_n\}$ in X is called a τ-Cauchy sequence in X if

$$\lim_{n,m\to\infty} F_{x_n - x_m}(t) = H(t) \text{ for all } t \geq 0.$$

The space X is said to be τ-complete if every τ-Cauchy sequence in X converges to a point in the topology τ.

DEFINITION 2.1. Let $(X,\widetilde{\mathcal{F}},\Delta)$ and $(Y,\mathcal{F},\Delta)$ be two Menger PN-spaces with the t-norm Δ satisfying the condition (2.1). Let τ_1 and τ_2 be the topologies induced by the family of neighborhoods of the type (2.2) on $(X,\widetilde{\mathcal{F}},\Delta)$ and $(Y,\mathcal{F},\Delta)$, respectively. A set-valued mapping $P: D(P) \subset X \to \Omega_Y$ (resp. a single-valued mapping $P: D(P) \subset X \to Y$) is said to be τ-closed if for any $x_n \in D(P)$ and $y_n \in P(x_n)$ (resp. $y_n = P(y_n)$), whenever $x_n \xrightarrow{\tau_1} x$ and $y_n \xrightarrow{\tau_2} y$, we have $x \in D(P)$ and $y \in P(x)$ (resp. $y = P(x)$).

DEFINITION 2.2. A function $\varphi: [0,+\infty) \to [0,+\infty)$ is said to satisfy the condition (Φ) if it is nondecreasing, $\varphi(0) = 0$ and

$$\lim_{n\to\infty} \varphi^n(t) = +\infty \text{ for all } t > 0. \tag{2.3}$$

REMARK 1. By Lemma 9.3.5 in [2], if φ satisfies the condition (Φ), then $\varphi(t) > t$ for all $t > 0$.

DEFINITION 2.3. A t-norm $\Delta: [0,1]\times[0,1] \to [0,1]$ is said to be of a h-type if for any $\lambda \in (0,1)$, there exists a number $\delta(\lambda) \in (0,1)$ such that, as $t > \delta(\lambda)$, the following holds uniformly

$$\Delta^k(t) > 1 - \lambda \text{ for all } k \geq 1,$$

where $\Delta^m(\cdot): [0,1] \to [0,1]$, $\Delta^1(t) = \Delta(t,t)$ and $\Delta^m(t) = \Delta(t, \Delta^{m-1}(t)) = \Delta(\Delta^{m-1}(t), t)$ for all $t \in (0,1)$ and $m = 2, 3, \cdots$.

REMARK 2. The t-norm Δ defined by $\Delta(a,b) = min\ \{a,b\}$ is an example of an h-type.

3. SET-VALUED AND SINGLE-VALUED OPERATOR EQUATIONS.

In this section, assume that $(X,\widetilde{\mathcal{F}},\Delta)$ is a τ_1-complete Menger PN-space, $(Y,\mathcal{F},\Delta)$ is a τ_2-complete Menger PN-space, Δ is a t-norm of h-type, and Ω_Y is a nonempty family of probabilistically bounded τ_2-closed subsets of Y.

Let $P: D(P) \subset X \to \Omega_Y$ and $P: D(P) \subset X \to Y$ be nonlinear set-valued and single-valued mappings, respectively, and $\Gamma: X \to L(Y,X)$, where $L(Y,X)$ denotes the space of all linear operators from Y into X. Let $\varphi: [0,+\infty) \to [0,+\infty)$ satisfy the condition (Φ) and $u \in Y$ be a given point. Then Γ is called a probabilistic contractor of a nonlinear set-valued mapping P (resp., a single-valued mapping P) with respect to u if, for all $x \in D(P)$ and $y \in \{y \in Y: x + \Gamma(x)y \in D(P)\}$,

$$F_{P(x+\Gamma(x)y),P(x)+y}(t) \geq min\{F_y(\varphi(t)), F_{P(x)-u}(\varphi(t)), F_{P(x+\Gamma(x)y)-u}(\varphi(t))\}, \tag{3.1a}$$

$$(resp.,\ F_{P(x+\Gamma(x)y)-P(x)-y}(t) \geq min\{F_y(\varphi(t)), F_{P(x)-u}(\varphi(t)), F_{P(x+\Gamma(x)y)-u}(\varphi(t))\}. \tag{3.1b}$$

for all $t \geq 0$.

REMARK 3. It follows from (4) of Lemma 2.1 that if Δ is a continuous t-norm with $\Delta(t,t) \geq t$ for all $t \in [0,1]$, then (3.1a) is equal to the following:

$$\begin{aligned} F_{P(x+\Gamma(x)y),P(x)+y}(t) \geq min\{&F_y(\varphi(t)), F_{P(x)-u}(\varphi(t)), \\ &F_{P(x+\Gamma(x)y)-u}(\varphi(t)), F_{P(x)+y-u}(2\varphi(t)), \\ &F_{P(x+\Gamma(x)y)-y-u}(2\varphi(t))\} \qquad \text{for all } t \geq 0. \end{aligned} \tag{3.2}$$

If $(Y,\mathcal{F},\Delta)$ is also a N.A. Menger PN-space, then (3.1a) is also equal to the following:

$$F_{P(x+\Gamma(x)y),P(x)+y}(t) \geq min\{F_y(\varphi(t)), F_{P(x)-u}(\varphi(t)), \tag{3.3}$$

$$F_{P(x+\Gamma(x)y)-u}(\varphi(t)), F_{P(x)+y-u}(\varphi(t)),$$

$$F_{P(x+\Gamma(x)y)-y-u}(\varphi(t)) \text{ for all } t \geq 0.$$

For the single-valued mapping P, we have the similar inequalities (3.2) and (3.3) which are equal to (3.1b).

Now we are ready to show the existence and uniqueness of solutions for the set-valued nonlinear operator equation

$$u \in P(x). \tag{3.4}$$

THEOREM 3.1. Let $(X, \widetilde{\mathcal{F}}, \Delta)$ be a τ_1-complete N.A. Menger PN-space, $(Y, \mathcal{F}, \Delta)$ be a τ_2-complete Menger PN-space and Δ be of a h-type. Let $P: D(P) \subset X \to \Omega_Y$ be a τ-closed set-valued mapping. Suppose that $\Gamma: X \to L(Y, X)$ satisfies the following conditions:

(1) $x + \Gamma(x)y \in D(P)$ for all $x \in D(P)$ and $y \in Y$;

(2) Γ is a probabilistic contractor of P with respect to u, i.e., Γ satisfies the condition (3.1a);

(3) There exists a constant $M > 0$ such that, for any $x \in D(P)$ and $y \in Y$,

$$\widetilde{F}_{\Gamma(x)y}(t) \geq F_y\left(\frac{t}{M}\right) \text{ for all } t \geq 0;$$

(4) For any $A, B \in \Omega_Y$ and $a \in A$, there exists a point $b \in B$ such that

$$F_{a-b}(t) \geq F_{A,B}(t) \text{ for all } t \geq 0.$$

Then the nonlinear set-valued operator equation (3.4) has a solution x^* in $D(P)$. Further, the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - \Gamma(x_n)y_n$$

converges to the solution x^* in the topology τ_1.

In order to prove Theorem 3.1, we need the following:

LEMMA 3.2. Let F_1 and F_2 be two distribution functions with $F_1(0) = F_2(0)$ and $\varphi: [0, +\infty) \to [0, +\infty)$ be a function satisfying the condition (Φ). If the following condition is satisfied:

$$F_1(t) \geq min\ \{F_1(\varphi(t)),\ F_2(\varphi(t)\} \text{ for all } t \geq 0, \tag{3.5}$$

then we have $F_2(\varphi(t)) \leq F_1(\varphi(t))$ for all $t \geq 0$.

PROOF. Suppose that the conclusion is not true. Then there exists a number $t_0 > 0$ such that

$$F_2(\varphi(t_o)) > F_1(\varphi(t_o)). \tag{3.6}$$

By (3.5), since $\varphi(t_o) > t_o$, we have $F_1(t_o) = F_1(\varphi(t_o))$. Let $t^* = max\{t > t_o: F_1(t)\ F_1(t_o)\}$. Since F_1 is left-continuous, such a t^* must exist and $t^* \geq \varphi(t_o)$. However, since $\varphi(t^*) > t^*$, we have

$$F_1(\varphi(t^*)) > F_1(t^*).$$

By the nondecreasing property of F_2 and (3.6), we have

$$F_2(\varphi(t^*)) \geq F_2(\varphi(t_o)) > F_1(\varphi(t_o)) = F_1(t^*).$$

Therefore, we have $F_1(t^*) < min\{F_1(\varphi(t^*)), F_2(\varphi(t^*))\}$, which is a contradiction. This completes the proof.

THE PROOF OF THEOREM 3.1.

(I) The case of $u = \theta$. In this case, (3.1a) can be written as follows:

$$F_{P(x+\Gamma(x)y), P(x)+y}(t) \geq min\{F_y(\varphi(t)), F_{P(x)}(\varphi(t)), F_{P(x+\Gamma(x)y)}(\varphi(t))\} \text{ for all } t \geq 0. \tag{3.7}$$

For any given $x_o \in D(p)$, take $y_o \in P(x_o)$ and let $x_1 = x_o - \Gamma(x_o)y_o$. By the assumption (1), we have $x_1 \in D(P)$. Replacing x and y by x_o and $-y_o$ in (3.7), respectively, from (4) of Lemma 2.1 and $\theta \in P(x_o) - y_o$, we have

$$\begin{aligned} F_{P(x_1)}(t) &\geq F_{P(x_1),\ P(x_o)-y_o}(t) \\ &= F_{P(x_o-\Gamma(x_o)y_o),\ P(x_o)-y_o}(t) \\ &\geq min\ \{F_{y_o}(\varphi(t)),\ F_{P(x_o)}(\varphi(t)),\ F_{P(x_1)}(\varphi(t))\} \\ &= min\ \{F_{y_o}(\varphi(t)),\ F_{P(x_1)}(\varphi(t))\} \end{aligned}$$

for all $t \geq 0$. By Lemma 3.2, we have

$$F_{y_o}(\varphi(t)) \leq F_{P(x_1)}(\varphi(t)) \tag{3.8}$$

for all $t \geq 0$. By the assumption (4), for $\theta \in P(x_o) - y_o$, there exist a point $y_1 \in P(x_1)$ such that

$$F_{y_1}(t) \geq F_{P(x_1), P(x_o)-y_o}(t)$$

for all $t \geq 0$. Hence, by (3.7) and (3.8), we have $F_{y_1}(t) \geq F_{y_o}(\varphi(t))$ for all $t \geq 0$. Let $x_2 = x_1 - P(x_1)y_1$. By the same method as stated above, there exists a point $y_2 \in P(x_2)$ such that

$$F_{y_2}(t) \geq F_{y_1}(\varphi(t)) \geq F_{y_o}(\varphi^2(t))$$

for all $t \geq 0$. Inductively, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = x_n - \Gamma(x_n)y_n; \tag{3.9}$$

$$y_n \in P(x_n); \tag{3.10}$$

$$F_{y_n}(t) \geq F_{y_o}(\varphi^n(t)). \tag{3.11}$$

for all $t \geq 0$. By the assumption (3), (3.9) and (3.11), we have

$$\widetilde{F}_{x_n - x_{n+1}}(t) = \widetilde{F}_{\Gamma(x_n)y_n}(t) \geq F_{y_n}\left(\frac{t}{M}\right) \geq \cdots \geq F_{y_o}\left(\varphi^n\left(\frac{t}{M}\right)\right)$$

for all $t \geq 0$. Hence, by $(PN-5)$, for any integers $m, n (m > n)$,

$$\begin{aligned} \widetilde{F}_{x_n - x_m}(t) &\geq \Delta(\widetilde{F}_{x_n - x_{n+1}}(t), \widetilde{F}_{x_{n+1} - x_m}(t)) \\ &\geq \Delta(\widetilde{F}_{x_n - x_{n+1}}(t), \Delta(\widetilde{F}_{x_{n+1} - x_{n+2}}(t), \end{aligned}$$

$$\overbrace{\Delta(\cdots,\Delta(}^{m-n-3}\widetilde{F}_{x_{m-2}-x_{m-1}}(t),\widetilde{F}_{x_{m-1}-x_m}(t))\cdots))$$

$$\geq \Delta\Big(F_{y_o}\Big(\varphi^n\Big(\frac{t}{M}\Big)\Big),\Delta\Big(F_{y_o}\Big(\varphi^{n+1}\Big(\frac{t}{M}\Big)\Big),$$

$$\overbrace{\Delta\Big(\cdots,\Delta\Big(}^{m-n-3}F_{y_o}\Big(\varphi^{m-2}\Big(\frac{t}{M}\Big)\Big),$$

$$F_{y_o}\Big(\varphi^{m-1}\Big(\frac{t}{M}\Big)\Big)\cdots\Big)\Big)$$

for all $t \geq 0$. Since φ satisfies the condition (Φ), $\varphi(t) > t$ and so, we have

$$\widetilde{F}_{x_n-x_m}(t) \geq \Delta\Big(F_{y_o}\Big(\varphi^n\Big(\frac{t}{M}\Big)\Big),\Delta\Big(F_{y_o}\Big(\varphi^n\Big(\frac{t}{M}\Big)\Big),$$

$$\overbrace{\Delta\Big(\cdots,\Delta\Big(}^{m-n-3}F_{y_o}\Big(\varphi^n\Big(\frac{t}{M}\Big)\Big),F_{y_o}\Big(\varphi^n\Big(\frac{t}{M}\Big)\cdots\Big)\Big)$$

$$= \Delta^{m-n-1}\Big(F_{y_o}\Big(\varphi^n\Big(\frac{t}{M}\Big)\Big)\Big)$$

for all $t \geq 0$. Since Δ is of a h-type, $\varphi^n\Big(\frac{t}{M}\Big) \to +\infty$ as $n \to \infty$ and so, for all $\lambda \in (0,1)$ and $t > 0$, there exists an integer $n(t,\lambda), n \geq n(t,\lambda), m > n$, such that

$$\widetilde{F}_{x_n-x_m}(t) \geq \Delta^{m-n-1}\Big(F_{y_o}\Big(\varphi^n\Big(\frac{t}{M}\Big)\Big)\Big) > 1-\lambda.$$

This means that the sequence $\{x_n\}$ is a τ-Cauchy sequence in X.
Since $(X,\widetilde{F},\Delta)$ is a τ_1-complete N.A. Menger PN-space, $x_n \xrightarrow{\tau_1} x^*$. By (2.3) and (3.11), we have

$$\lim_{n\to\infty} F_{y_n}(t) = 0 \text{ for all } t > 0,$$

i.e., $y_n \xrightarrow{\tau_2} \theta$. Therefore, from the τ-closedness of P and (3.10), we have $x^* \in D(P)$ and $\theta \in P(x^*)$, i.e., x^* is a solution of (3.4).

(II). The case of $u \neq \theta$. Let $T(x) = P(x) - u$ for $x \in D(P)$. Then $D(P) = D(T)$ and P satisfying (3.1a) is equal to T satisfying (3.7). Therefore, by using the case of $u = \theta$, we can show the existence of solution for the nonlinear set-valued operator equation $\theta \in T(x)$. This completes the proof.

For the nonlinear single-valued operation equation

$$u = P(x), \tag{3.12}$$

we also have the following:

THEOREM 3.3. Let $(X,\widetilde{\mathcal{F}},\Delta)$, $(Y,\mathcal{F},\Delta)$ and Δ be as in Theorem 3.1. Let $P: D(P) \subset X \to Y$ be a τ-closed single-valued operator and $\Gamma: X \to L(Y,X)$ be such that

(1) $x + \Gamma(x)y \in D(P)$ for all $x \in D(P)$ and $y \in Y$;

(2) Γ is a probabilistic contractor of P with respect to u, i.e., Γ satisfies the condition (3.1b);

(3) There is a constant $M > 0$ such that for any $x \in D(P)$ and $y \in Y$,

$$\widetilde{F}_{\Gamma(x)y}(t) \geq F_y\Big(\frac{t}{M}\Big) \text{ for all } t \geq 0.$$

Then the equation (3.12) has a solution x^* in $D(P)$ and for any given $x_o \in D(P)$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - \Gamma(x_n)(P(x_n) - u) \tag{3.13}$$

converges to the solution x^* of equation (3.12) in the topology τ_1.
If $\Gamma(x^*): Y \to X$ is surjective, then x^* is the unique solution of (3.12).

PROOF. Without loss of generality, we may assume that $u = \theta$. In this case, (3.1b) can be written as follows:

$$F_{P(x+\Gamma(x)y) - P(x) - y)}(t) \geq min\{F_y(\varphi(t)), F_{P(x)}(\varphi(t)), F_{P(x+\Gamma(x)y)}(\varphi(t))\} \tag{3.14}$$

for all $t \geq 0$. By the condition (1) and (3.13), we have $x_n \in D(P)$ for $n = 0, 1, 2, \cdots$. Replacing x and y by x_n and $-P(x_n), n = 0, 1, 2, \cdots$, in (3.14), respectively, we have

$$F_{P(x_{n+1})}(t) \geq min\{F_{P(x_n)}(\varphi(t)), F_{P(x_n)}(\varphi(t)), F_{P(x_{n+1})}(\varphi(t))\}$$

$$= min\{F_{P(x_n)}(\varphi(t)), F_{P(x_{n+1})}(\varphi(t))\}$$

for all $t \geq 0$. By Lemma 3.2, we have

$$F_{P(x_{n+1})}(t) \geq F_{P(x_n)}(\varphi(t)) \geq \cdots \geq F_{P(x_o)}(\varphi^{n+1}(t))$$

for all $t \geq 0$. In view of the assumption (3), (3.13) and (3.15), we have

$$\widetilde{F}_{x_n - x_{n+1}}(t) = \widetilde{F}_{\Gamma(x_n)(P(x_n))}(t) \geq F_{P(x_n)}\left(\frac{t}{M}\right) \geq \cdots \geq F_{P(x_o)}\left(\varphi^n\left(\frac{t}{M}\right)\right)$$

for all $t \geq 0$. By the same method as in the proof of Theorem 3.1, we can prove that $\{x_n\}$ is a τ_1-Cauchy sequence in X. Since $(X, \widetilde{\mathfrak{F}}, \Delta)$ is τ-complete, $x_n \xrightarrow{\tau_1} x^*$. Hence, from (2.3) and (3.15), we have $P(x_n) \xrightarrow{\tau_2} \theta$. Therefore, by the τ-closedness of P, we have $x^* \in D(P)$ and $P(x^*) = \theta$.

Next, we prove the uniqueness of solution of the operator equation $u = P(x)$. In fact, if $x^{**} \in D(P)$ and $P(x^{**}) = \theta$, by the surjectivity of $\Gamma(x^*)$, there exists a point $y \in Y$ such that $x^{**} - x^* = \Gamma(x^*)y$. Since $P(x^*) = P(x^{**}) = \theta$ and $F_\theta(t) = H(t)$, from (3.14), we have

$$F_y(t) = F_{P(x^{**}) - P(x^*) - y}(t) \geq min\ \{F_y(\varphi(t)),\ F_{P(x^*)}(\varphi(t)),\ F_{P(x^{**})}(\varphi(t))\}$$

$$= F_y(\varphi(t))$$

for all $t \geq 0$, which implies that

$$F_y(t) \geq F_y(\varphi(t)) \geq \cdots \geq F_y(\varphi^n(t))$$

for all $t \geq 0$ and $n = 1, 2, \cdots$. Letting $n \to \infty$, from (2.3) we have $F_y(t) = 1$ for all $t \geq 0$. This means that $y = \theta$, i.e., $x^* = x^{**}$. This completes the proof.

4. FIXED POINT THEOREMS.

In this section, using Theorems 3.1 and 3.2, we can obtain two fixed point theorems for set-valued and single-valued mappings:

THEOREM 4.1. Let $(X, \mathfrak{F}, \Delta)$ be a τ-complete N.A. Menger PN-space and Δ be a t-norm of a h-type. let $T: X \to \Omega_x$ satisfy the following condition:

$$F_{Tx, Ty}(t) \geq min\{F_{x-y}(\varphi(t)), F_{x-Tx}(\varphi(t)), F_{y-Ty}(\varphi(t))\} \tag{4.1}$$

for all $t \geq 0$ and $x, y \in X$, where $\varphi: [0,\infty) \to [0,\infty)$ satisfies the condition (Φ). Suppose further that, for any $A, B \in \Omega_X$ and $a \in A$, there exists a point $b \in B$ such that

$$F_{a-b}(t) \geq F_{A,B}(t) \text{ for all } t \geq 0.$$

Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$, i.e., x^* is a fixed point of T.

PROOF. Putting $P(x) = x - Tx$ and $\Gamma(x) = I_X$ (:the identity mapping on X), the mappings P and Γ satisfy all the hypotheses of Theorem 3.1. Therefore, there exists a point $x^* \in X$ such that $\theta \in P(x^*) = x^* - Tx^*$, which means that x^* is a fixed point of T. This completes the proof.

THEOREM 4.2. Let $(X, \mathcal{F}, \Delta)$ be a τ-complete Menger PN-space and Δ be a t-norm of a h-type. Let $T: X \to X$ satisfy the following condition:

$$F_{Tx-Ty}(t) \geq min\{F_{x-y}(\varphi(t)), F_{x-Tx}(\varphi(t)), F_{y-Ty}(\varphi(t))\}$$

for all $t \geq 0$ and $x, y \in X$, where $\varphi: [0,\infty) \to [0,\infty)$ satisfies the conditions (Φ).
Then there exists a point $x^* \in X$ such that $x^* = Tx^*$, that is, x^* is a unique fixed point of T and, for any $x_o \in X$, the iterative sequence $\{x_n\}$ in X converges to x^* in the topology τ, where $x_n = Tx_{n-1}, n = 2, 3, 4, \cdots$.

PROOF. Putting $P(x) = x - Tx$ and $\Gamma(x) = I_X$, the mappings P and Γ satisfy all the hypotheses of Theorem 3.2. Therefore, there exists a point $x^* \in X$ such that $\theta = P(x^*) = x^* - Tx^*$, i.e., x^* is a fixed point of T. This completes the proof.

REMARK 4. (1) In Theorem 4.2, if we assume that $\Delta(t,t) \geq t$ for all $t \in [0,1]$, then by Remark 3, (4.1) can be weakened as follows:

$$F_{Tx-Ty}(t) \geq min\{F_{x-y}(\varphi(t)), F_{x-Tx}(\varphi(t)), F_{y-Ty}(\varphi(t)), F_{y-Tx}(\varphi(t)), F_{x-Ty}(\varphi(t))\}$$

for all $t \geq 0$.

(2) Theorem 4.2 extends fixed point theorems of Chang [3] and others.

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