CHARACTERISTIC POLYNOMIALS OF SOME WEIGHTED GRAPH BUNDLES AND ITS APPLICATION TO LINKS

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ABSTRACT. In this paper, we introduce weighted graph bundles and study their characteristic polynomial. In particular, we show that the characteristic polynomial of a weighted $K_2$ ($K_2$)-bundles over a weighted graph $\Gamma_\omega$ can be expressed as a product of characteristic polynomials two weighted graphs whose underlying graphs are $\Gamma$. As an application, we compute the signature of a link whose corresponding weighted graph is a double covering of that of a given link.

KEY WORDS AND PHRASES. Graphs, weighted graphs, graph bundles, characteristic polynomials, links, signature.

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1. INTRODUCTION.

Let $\Gamma$ be a simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Let $R$ be the field of real numbers. A weighted graph is a pair $F(r, w)$, where $F$ is a graph and $w: V(F) \cup E(F) \rightarrow R$ is a function. We call $F$ the underlying graph of $F$, and $w$ the weight function of $F$. In particular, if $w(E(r)) \subseteq \{0, 1\}$ and $w(V(r)) = \{0\}$, then we call $F$ a signed graph.

Let $V(\Gamma) = \{u_1, \ldots, u_n\}$. The adjacency matrix of $\Gamma_\omega$ is an $n \times n$ matrix $A(\Gamma_\omega) = (a_{ij})$ defined as follows:

$$a_{ij} = \begin{cases} 
\omega(e) & \text{if } e = u_iu_j \in E(\Gamma) \text{ and } i \neq j, \\
\omega(u_i) & \text{if } i = j, \\
0 & \text{otherwise,}
\end{cases}$$

for $1 \leq i, j \leq n$.

The characteristic polynomial $P(\Gamma_\omega; \lambda) = |\lambda I - A(\Gamma_\omega)|$ of the adjacency matrix $A(\Gamma_\omega)$ is called the characteristic polynomial of the weighted graph $\Gamma_\omega$. A root of $P(\Gamma_\omega; \lambda)$ is called an eigenvalue of $\Gamma_\omega$.

Note that if the weight function $L$ of $\Gamma$ is defined by $L(e) = -1$ for $e \in E(\Gamma)$ and $L(u) = \deg(u)$ for $u \in V(\Gamma)$, where $\deg(u)$ denotes the degree of $u$, that is, the number of edges incident to $u$, then the weighted adjacency matrix $A(\Gamma_L)$ is called the Laplacian matrix of $\Gamma$. We call $L$ the Laplacian function of $\Gamma$. The number of spanning trees of a connected graph $\Gamma$ is the
value of any cofactor of $A(\Gamma_L^\phi)$ [Matrix tree theorem] and is equal to the value $\frac{1}{n!} \prod_{\lambda \neq 0} \lambda$, where $\lambda$ runs through all non-zero eigenvalues of $A(\Gamma_L^\phi)$. Moreover, the eigenvalues of $A(\Gamma_L^\phi)$ may be used to calculate the radius of gyration of a Gaussian molecule. For more applications of the eigenvalues of $A(\Gamma_L^\phi)$, the reader is suggested to refer [5].

2. WEIGHTED GRAPH BUNDLES.

First, we introduce a weighted graph bundle. Every edge of a graph $\Gamma$ gives rise to a pair of oppositely directed edges. We denote the set of directed edges of $\Gamma$ by $D(\Gamma)$. By $e^{-1}$ we mean the reverse edge to an edge $e \in D(\Gamma)$. For any finite group $G$, a $G$-voltage assignment of $\Gamma$ is a function $\phi: D(\Gamma) \rightarrow G$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in D(\Gamma)$. We denote the set of all $G$-voltage assignments of $\Gamma$ by $C^1(\Gamma; G)$. Let $\Lambda$ be another graph and let $\phi \in C^1(\Gamma; \text{Aut}(\Lambda))$, where $\text{Aut}(\Lambda)$ is the group of all graph automorphisms of $\Lambda$. Now, we construct a graph $\Gamma \times ^\phi \Lambda$ as follows: $V(\Gamma \times ^\phi \Lambda) = V(\Gamma) \times V(\Lambda)$. Two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $\Gamma \times ^\phi \Lambda$ if either $u_1 u_2 \in D(\Gamma)$ and $v_1 v_2 = \phi(u_1 u_2) v_1$ or $u_1 = u_2$ and $v_1 v_2 \in E(\Lambda)$. We call $\Gamma \times ^\phi \Lambda$ the $\Lambda$-bundle over $\Gamma$ associated with $\phi$ and the natural map $p^\phi: \Gamma \times ^\phi \Lambda \rightarrow \Gamma$ the bundle projection. We also call $\Gamma$ and $\Lambda$ the base and the fibre of $\Gamma \times ^\phi \Lambda$, respectively. Note that the map $p^\phi$ maps vertices to vertices but an image of an edge can be either an edge or a vertex. If $\Lambda$ is the complement $\bar{K}_n$ of the complete graph $K_n$ of $n$ vertices, then every $\Lambda$-bundle over $\Gamma$ is an $n$-fold covering graph of $\Gamma$.

Let $\Gamma_\omega$ and $\Lambda_\mu$ be two weighted graphs and let $\phi \in C^1(\Gamma; \text{Aut}(\Lambda))$. We define the product of $\mu$ and $\omega$ with respect to $\phi$, $\omega \times ^\phi \mu$, as follows:

1. For each vertex $(u, v)$ of $V(\Gamma \times ^\phi \Lambda), (\omega \times ^\phi \mu)(u, v) = \omega(u) + \mu(v)$.
2. For each edge $e = (u_1, v_1)(u_2, v_2)$ of $E(\Gamma \times ^\phi \Lambda)$,

$$(\omega \times ^\phi \mu)(e) = \begin{cases} \omega(u_1 u_2) & \text{if } u_1 u_2 \in D(\Gamma) \text{ and } v_1 v_2 = \phi(u_1 u_2) v_1 \\ \mu(v_1 v_2) & \text{if } u_1 = u_2 \text{ and } v_1 v_2 \in E(\Gamma). \end{cases}$$

We call the weighted graph $(\Gamma \times ^\phi \Lambda)_{\omega \times ^\phi \mu}$ the $\Lambda_\mu$-bundle over $\Gamma_\omega$ associated with $\phi$. Briefly, we call it a weighted graph bundle.

3. CHARACTERISTIC POLYNOMIALS.

In this section, we give a computation for the characteristic polynomial of a weighted graph bundle $\Gamma \times ^\phi \Lambda$, where $\Lambda$ is either complete graph $K_2$ of two vertices or its complement $\bar{K}_2$, and study their related topics. Note that $\text{Aut}(K_2) = \text{Aut}(\bar{K}_2) = Z_2$.

For a given graph $\Gamma$ with weight function $\omega$ and for a $\phi \in C^1(\Gamma; Z_2)$, we define a new weight function $\omega^\phi$ on $\Gamma$ as follows:

**FIGURE 1.** The graphs $C_4 \times ^\phi K_2$ and $(C_4 \times ^\phi K_2)_{\omega \times ^\phi \mu}$. 

For $e \in E(\Gamma)$,
\[
\omega^\phi(e) = \begin{cases} 
\omega(e) & \text{if } \phi(e) = 1 \\
-\omega(e) & \text{if } \phi(e) = -1
\end{cases}
\]

For $v \in V(\Gamma)$, $\omega^\phi(v) = u(v)$.

A subgraph of $\Gamma$ is called an elementary configuration if its components are either complete graph $K_1$ or $K_2$ or a cycle $C_m (m \geq 3)$. We denote by $E_k$ the set of all elementary configurations of $\Gamma$ having $k$ vertices. In [3], the characteristic polynomial of a weighted graph $\Gamma_\omega$ is given as follows:

\[
P(\Gamma_\omega; \lambda) = \sum_{k=0}^{n} a_k(\Gamma_\omega)\lambda^{n-k},
\]

where

\[
a_k(\Gamma_\omega) = \sum_{S \in E_k} (-1)^{\kappa(S)} \left| C(S) \right| \prod_{u \in I_u(S)} \omega(u) \prod_{e \in I_e(S)} \omega(e) \prod_{c \in C(c)} \omega(c).
\]

In the above equation, symbols have the following meaning: $\kappa(S)$ is the number of components of $S$, $C(S)$ the set of all cycles, $C_m (m \geq 3)$, in $S$, and $I_u(S)$\(\cap I_e(S)\) is the set of all isolated vertices (edges) in $S$. Moreover, the product over empty index set is defined to be 1.

For a fixed voltage assignment $C(\Gamma; \mathbb{Z})$, we denote by $E_\phi$ the set of edges of $\Gamma$ such that $\phi(e) = -1$, i.e., $E_\phi = \{ e \in E(\Gamma) : \phi(e) = -1 \}$. Let $\Gamma(E_\phi)$ be the edge subgraph of $\Gamma$ induced by $E_\phi$ having weight zero in vertices. If $\Gamma_\omega$ is a weighted graph, then the weight function of its subgraph $S$ is the restriction of $\omega$ on $S$.

**Theorem 1.** Let $K_2$ be a constant weighted graph, say $\mu(v) = c$ for $v \in K_2$. Then, for each $\phi \in C^1(\Gamma; \mathbb{Z})$, we have

\[
P((\Gamma K_2)_{\omega,^\phi, c}; \lambda) = P(\Gamma_\omega; \lambda - c)P(\Gamma_\omega; \lambda - c).
\]

**Proof.** Let $A(\Gamma_\omega)$ be the adjacency matrix of $\Gamma_\omega$ and let $A(\Gamma_{\omega,^\phi})$ the adjacency matrix of $\Gamma_{\omega,^\phi}$. Then we have

\[
A(\Gamma_\omega) = A(\Gamma \setminus (E_{\phi, -1})_{\omega}) + A(\Gamma(E_{\phi, -1})_{\omega}),
\]

\[
A(\Gamma_{\omega,^\phi}) = A(\Gamma \setminus (E_{\phi, -1})_{\omega}) - A(\Gamma(E_{\phi, -1})_{\omega}).
\]

Let $V(\Gamma K_2) = \{(u_1, 1), \cdots , (u_n, 1), (u_1, -1), \cdots , (u_n, -1)\}$. If is not difficult to show that

\[
A(\Gamma K_2)_{\omega,^\phi, c} = A(\Gamma_\omega) - A(\Gamma(E_{\phi, -1})_{\omega}) + \begin{bmatrix} c & 0 \\ 0 & \ddots \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
+ (A(\Gamma(E_{\phi, -1})_{\omega})) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

Let $M$ be a regular matrix of order 2 satisfying

\[
M^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

Put

\[
X = A(\Gamma_\omega) - A(\Gamma(E_{\phi, -1})_{\omega}) + \begin{bmatrix} c & 0 \\ 0 & \ddots \end{bmatrix}
\]

\[
Y = A(\Gamma(E_{\phi, -1})_{\omega}).
\]
Then

\[
(I \otimes M^{-1}) A \left( (\Gamma \times \phi K_2)^{\omega \times \phi_c} \right) (I \otimes M)
\]

\[
= \begin{bmatrix}
X + Y & 0 \\
0 & X - Y
\end{bmatrix}
\]

\[
A(\Gamma \omega) + \begin{bmatrix}
c & 0 \\
0 & c
\end{bmatrix}
\]

\[
A(\Gamma \omega) + \begin{bmatrix}
c & 0 \\
0 & c
\end{bmatrix}
\]

Since \(|(I \otimes M^{-1})(I \otimes M)| = 1\) and

\[
\left| I - A \left( (\Gamma \times \phi K_2)^{\omega \times \phi_c} \right) \right| = \left| I - (I \otimes M^{-1}) A \left( (\Gamma \times \phi K_2)^{\omega \times \phi_c} \right) (I \otimes M) \right|
\]

we have our theorem. \(\square\)

**THEOREM 2.** Let \(K_2^\mu = (K_2, \mu)\) be a weighted graph having constant weight on vertices. Then, for each \(b \in C(\Gamma; Z_2)\), we have

\[
P((\Gamma \times \phi K_2)^{\omega \times \phi_c}; \lambda) = P(\Gamma \omega; \lambda - c_v - c_e)P(\Gamma \omega; \lambda - c_v + c_e),
\]

where \(c_v = \mu(v_1) = \mu(v_2)\) for the vertices \(v_1, v_2\) and \(c_e = \mu(e)\) for the edge \(e\) in \(K_2\).

**PROOF.** Clearly, we have

\[
A((\Gamma \times \phi K_2)^{\omega \times \phi}) = (A(\Gamma \omega) - A(\Gamma(E_{\phi = 1})\omega) + \begin{bmatrix}
c_v & 0 \\
0 & c_v
\end{bmatrix}) \otimes \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
+ \left( A(\Gamma(E_{\phi = 1})\omega) + \begin{bmatrix}
c_e & 0 \\
0 & c_e
\end{bmatrix} \right) \otimes \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

where \(c_v = \mu(v_1) = \mu(v_2)\) and \(c_e = \mu(e)\) for the edge \(e\) in \(K_2\). Let \(M\) be a regular matrix of order 2 satisfying

\[
M^{-1} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} M = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

Then

\[
(I \otimes M^{-1}) A \left( (\Gamma \times \phi K_2)^{\omega \times \phi_c} \right) (I \otimes M)
\]

\[
= \begin{bmatrix}
X + Y + \begin{bmatrix}
c_e & 0 \\
0 & c_e
\end{bmatrix} & 0 \\
0 & X - Y + \begin{bmatrix}
c_e & 0 \\
0 & c_e
\end{bmatrix}
\end{bmatrix}
\]
where $X$ and $Y$ are the same matrices as in the proof of Theorem 1 and for $i = 1, 2$.

Using method similar to the proof of Theorem 1, we have our theorem.

Note that for any $\phi \in C^1(\Gamma; \text{Aut}(\Lambda))$, the Laplacian function of $\Gamma \times \phi \Lambda$ is the product of Laplacian functions of $\Gamma$ and $\Lambda$ with respect to $\phi$. Clearly, the Laplacian function of the $K_2$ is the zero function; and the Laplacian function of the $K_2$ has value 1 and $-1$ for each of its vertices and its edge, respectively. We shall denote the Laplacian function of a graph by $L$ if it makes no confusion. Then Theorem 1 and Theorem 2 give the following corollary.

**COROLLARY 1.** For any $\phi \in C^1(\Gamma; Z)$,

1. $P((\Gamma \times \phi K_2); \lambda) = P(\Gamma; \lambda)P(\Gamma_\phi; \lambda)$.
2. $P((\Gamma \times \phi K_2); \lambda) = P(\Gamma; \lambda)P(\Gamma_\phi; \lambda - 2)$.

Now, we consider another invariant of weighted graphs called the signature. Since $A(\Gamma_\omega)$ is symmetric, $A(\Gamma_\omega)$ can be diagonalized through congruence over $R$. Let $d_+$ denote the number of positive diagonal entries, and $d_-$ the number of negative diagonal entries. The signature of a weighted graph $(\Gamma_\omega)$ is defined by $\sigma(\Gamma_\omega) = d_+ - d_-$ and is denoted by $\sigma(\Gamma_\omega)$. It is an invariant for weighted 2-isomorphic graphs (see [7]).

From now on, we will consider the weight function on $K_2$ as zero function and the weight function $\mu$ on $K_2$ as the map defined by $\mu(v) = 0$ for each $v \in V(K_2)$ and $\mu(e) = c_e$ for the edge $e$ of $K_2$. Then we can compute the signature of a double covering of $\Gamma$.

**COROLLARY 2.** $\sigma((\Gamma \times \phi K_2); \lambda, \phi) = \sigma(\Gamma_\omega) + \sigma(\Gamma_\phi)$ for $\phi \in C^1(\Gamma; Z)$.

For convenience, we adapt the following notations. For a real number $c$, a weighted graph $\Gamma_\eta$ and an eigenvalue $\lambda$ of $\Gamma_\eta$,

- $P(c_{\eta}) = \{ \lambda < 0: \lambda + c > 0 \}$,
- $P(c_{\eta})^+ = \{ \lambda > 0: \lambda + c > 0 \}$,
- $Z(c_{\eta}) = \{ \lambda \neq 0: \lambda + c = 0 \}$,
- $N(c_{\eta}) = \{ \lambda < 0: \lambda + c < 0 \}$,
- $N(c_{\eta})^+ = \{ \lambda > 0: \lambda + c < 0 \}$.

We also denote the multiplicity of $\lambda$ by $m_{\eta}(\lambda)$.

By using the above notations and Theorem 2, we get the signature of a $K_2$-bundle over $\Gamma$.

**COROLLARY 3.** For $\phi \in C^1(\Gamma; Z)$,

1. if $c_e \geq 0$, then

$$\sigma((\Gamma \times \phi K_2); \lambda, \phi) = \sigma(\Gamma_\omega) + \sigma(\Gamma_\phi)$$

$$+ \left( 2 \sum_{n \in P(c_e)} m_{\omega}(\lambda) + m_{\phi}(0) + \sum_{\lambda \in Z(c_e)} m_{\omega}(\lambda) \right)$$

$$- \left( 2 \sum_{n \in N(-c_e)} m_{\omega}(\lambda) + m_{\phi}(0) + \sum_{\lambda \in Z(-c_e)} m_{\omega}(\lambda) \right).$$
(2) if $c_r < 0$, then
\[
\sigma(\Gamma \times \Phi K_i) \omega = \sigma(\Gamma) + \sigma(\Gamma)_{\phi} - \left(2 \sum_{\lambda \in \Pi_{\Omega}} m_\omega(\lambda) + m_\omega(0) + \sum_{\lambda \in \Delta} \omega(\epsilon(\lambda)) m_\omega(\lambda)\right)
\]
\[
+ \left(2 \sum_{\lambda \in \Pi(-C_i)} m_\omega(\lambda) + m_\omega(0) + \sum_{\lambda \in \Delta} \omega(\epsilon(\lambda)) m_\omega(\lambda)\right)
\]

**Remark.** Though the results in this section stated only for a simple graph, it remains true for any graph.

4. **Applications to Links.**

In a signed graph $\Gamma$, an edge $e$ of $\Gamma$ is said to be positive if $\omega(e) = 1$ and negative otherwise. For a signed graph $\Gamma$, we define a new weight function $\tilde{\omega}$ of $\Gamma$ by $\tilde{\omega}(e) = \omega(e)$ for any edge $e \in \Gamma$ and $\tilde{\omega}(u_i) = \sum_{i=1, i \neq j}^n a_{ij}$, where $a_{ij}$ is the number of positive edges minus the number of negative edges which have two end vertices $u_i$ and $u_j$. Given a knot or link $L$ in $\mathbb{R}^3$, we project it into $\mathbb{R}^2$ so that each crossing point has proper double crossing. The image of $L$ is called a **link (or knot) diagram** of $L$, and we do not distinguish between a diagram and the image of $L$.

We may assume without loss of generality that a link diagram $\tilde{L}$ of $L$ intersects itself transversely and has only finitely many crossings. The link diagram $\tilde{L}$ divides $\mathbb{R}^2$ into finitely many domains, which will be classified as shaded or unshaded. No two shaded or unshaded domains have an edge in common. We now construct a signed planar graph $\Gamma$ from $\tilde{L}$ as follows: take a point $v_i$ from each unshaded domain $D_i$. These points form the set of vertices $V(\Gamma)$ of $\Gamma$. If the boundaries of $D_i$ and $D_j$ intersect $k$-times, say, crossing at $c_{e_1}, c_{e_2}, \ldots, c_{e_k}$, then we form multiple edges $e_{e_1} c_{e_2} \ldots c_{e_k}$ on $\mathbb{R}^2$ with common end vertices $v_i$ and $v_j$, where each edge $e_{e_m}$ passes through a crossing $c_{e_m}$ for $m = 1, 2, \ldots, k$. To define the weight of an edge, first, we define the index $\epsilon(c)$ to each crossing $c$ of the link diagram as in Figure 2. To each edge of $\Gamma$ passes through exactly one crossing, say $c$, of $\tilde{L}$, the weight $\omega(e)$ will be defined as $\omega(e) = \epsilon(c)$. (See Figure 3.)

![Figure 2. The index \( \epsilon(c) \).](image-url)
SOME WEIGHTED GRAPH BUNDLES

FIGURE 3. The correspondence between \( \mathcal{L} \) and \( \Gamma_w(\mathcal{L}) \).

The resulting signed planar is called the graph of a link with respect to \( \mathcal{L} \), and it is denoted by \( \Gamma_w(\mathcal{L}) \). The signed planar graph \( \Gamma_w(\mathcal{L}) \) depends not only on \( \mathcal{L} \) but also on shading. Conversely, given a signed planar graph \( \Gamma_\rho \), one can construct uniquely the link diagram \( L(\Gamma_\rho) \) of a link so that \( \Gamma_w(L(\Gamma_\rho)) = \Gamma_\rho \).

FIGURE 4. The index \( \omega(c) \).

Suppose that we are given an oriented link \( L \). The orientation of \( L \) induces the orientation of a diagram \( \mathcal{L} \). We then define the second index \( \omega(c) \), called the twist or writhe at each crossing \( c \) as shown in Figure 4. We now need the third index \( \eta_\rho(c) \) at each crossing \( c \). Let \( \mathcal{L} \) be an oriented diagram and \( \rho \) shading on \( \mathcal{L} \). Let \( \eta_\rho(c) = \omega(c)\delta_{\delta(c),\omega(c)} \), where \( \delta \) denotes Kronecker's delta. We define \( \eta_\rho(\mathcal{L}) = \sum \eta_\rho(c) \), where the summation runs over all crossing in \( \mathcal{L} \). The index \( \eta_\rho(\mathcal{L}) \) depends not only on the shading \( \rho \) but also on the orientation of \( \mathcal{L} \). The following Lemma can be found in ([7], [4]).

**LEMMA 1.** The signature \( \sigma(L) \) of a link \( L \) is \( \sigma(L) = \sigma(\Gamma(\mathcal{L})) - \eta_\rho(\mathcal{L}) \).

Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be link diagrams of \( L_1 \) and \( L_2 \), respectively. The link \( L_2 \) is called a double covering of the link \( L_1 \) if \( \Gamma_w(\mathcal{L}_2) \) is a double covering of \( \Gamma_w(\mathcal{L}_1) \) as weighted graphs and it can be extended to a branched covering on \( R^2 \). Let \( \phi \) be a voltage assignment in \( C^1(\Gamma_w(\mathcal{L}_1); \mathbb{Z}_2) \) such that \( \phi(e) = -1 \) for some edge \( e \) and \( \phi(e) = 1 \) otherwise, then \( \Gamma_w(\mathcal{L}) \times \times K_2 \) is a planar double covering of \( \Gamma_w(\mathcal{L}) \) of which the corresponding link is a double covering of \( L \).

Therefore, one can construct the double covering link diagram \( \mathcal{L} (\Gamma_w(\mathcal{L})) \times K_2 \) of \( \mathcal{L} \). Moreover, we can give an orientation on \( \mathcal{L} (\Gamma_w(\mathcal{L})) \times K_2 \) so that the covering map from \( \mathcal{L} \) to \( \mathcal{L} (\Gamma_w(\mathcal{L})) \times K_2 \) preserves the orientation. We have \( \eta_\rho(\mathcal{L} (\Gamma_w(\mathcal{L})) \times K_2) = 2\eta_\rho(\mathcal{L}) \) (see Figure 5).
Therefore, by using Lemma and Corollary 2, we get the following theorem.

**THEOREM 3.** For any oriented link diagram $\Gamma$, 

$$\sigma(\bar{\Gamma}(\Gamma) \times \ast K_2) = \sigma(\Gamma) + \sigma(\bar{\Gamma}) - 2\eta(\bar{\Gamma})$$

for each $\phi \in \mathcal{C}^1(\Gamma; Z_2)$ such that $\phi(e) = -1$ for some edge $e \in \Gamma$ and $\phi(e) = 1$ otherwise.

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