CR-HYPERSURFACES OF COMPLEX PROJECTIVE SPACE

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ABSTRACT. We consider compact n -dimensional minimal foliate CR -real submanifolds of a complex projective space. We show that these submanifolds are great circles on a 2-dimensional sphere provided that the square of the length of the second fundamental form is less than or equal to $n-1$.

KEY WORDS AND PHRASES. Kaehler manifold, CR -submanifold, mixed foliate, hypersurfaces of complex projective space.

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1. INTRODUCTION.

CR -submanifolds of a Kaehlerian manifold have been defined by A. Bejancu [1]. These manifolds have then been studied by several authors. Among these are B.Y. Chen [2],[3], K. Yano, M. Kon, K. Sekigawa, and A. Ross [4].

In particular CR -submanifolds isometrically immersed in complex projective space have been considered by K. Yano and M. Kon [6]. They studied CR -submanifolds isometrically immersed in complex projective space with geometric properties such as semi-flat normal connection or parallel mean curvature. In this paper we consider minimal proper CR -hypersurfaces of a complex projective space. for such submanifolds we have obtained the following:

THEOREM 1. Let M be a compact n -dimensional minimal foliate CR -real hypersurface of a complex projective space. If the square of the length of the second fundamental form is $\leq (n-1)$, then M is a totally real submanifold of dimension 1. In fact M is a great circle on S^2 .

2. PRELIMINARIES.

A submanifold M of a Kaehler manifold is called a CR -submanifold if there is a differentiable distribution $D: x \rightarrow D_x \subseteq T_x M$ on M satisfying the following conditions:

- D is holomorphic i.e., $JD = D$ for each $x \in M$, where J is the almost complex structure.
- The complementary orthogonal distribution $D^\perp: x \rightarrow D_x^\perp \subseteq T_x M$ is totally real i.e., $J D^\perp \subseteq T_x M$ where $T_x M$ is the normal bundle. If $\dim D_x^\perp = 0$ (respectively, $\dim D_x = 0$), M is called a complex (respectively totally real) submanifold. A CR -submanifold is said to be proper if it is neither complex nor totally real. The normal bundle $T_x M$ splits as $T_x M = J D^\perp \oplus \mu$, where μ is invariant sub-bundle of $T_x M$ under J .

Now let $\bar{M}$ be the complex projective space, which is a Kaehler manifold with constant holomorphic sectional curvature 4. Let g be the Hermitian metric tensor field of $\bar{M}$. Suppose that M is an n -dimensional CR -hypersurface of $\bar{M}$. We denote by the same g the Riemannian metric tensor field induced on M from that of $\bar{M}$. Let ∇ , $\bar{\nabla}$, $\bar{\nabla}^\perp$ be the Riemannian connections on M , $\bar{M}$ and the normal bundle respectively. Then we have Gauss formula and Weingarten formula;

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (2.1)$$

$$\bar{\nabla}_X N = -A_N X, \quad N \in \bar{T}^\perp M \quad (2.2)$$

where $h(X, Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X, Y), N) = g(A_N X, Y) \quad (2.3)$$

where X and Y are vector fields on M .

We also have the following Gauss equation

$$\begin{aligned} R(X, Y; Z, W) &= g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(JY, Z)g(JX, W) - g(JX, Z)g(JY, W) \\ &\quad + 2g(X, JY)g(JZ, W) + g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)) \end{aligned} \quad (2.4)$$

where $R(X, Y; Z, W)$ is the Riemannian curvature tensor of type $(0, 4)$.

Let $H = \frac{1}{n}$ (trace h) be the mean curvature vector. Then M is said to be minimal if $H = 0$.

A CR -submanifold is said to be mixed foliate if

(a) the holomorphic distribution D is integrable.

(b) $h(X, \xi) = 0$ for $X \in D$ and $\xi \in \bar{D}$.

For mixed foliate submanifolds of a complex space form $\bar{M}(c)$ (i.e., a Kaehler manifold of constant holomorphic sectional curvature c), the following result is well known

THEOREM 2.[3] If M is a mixed foliate proper CR -submanifold of a complex space form $\bar{M}(c)$, then we have $c \leq 0$.

3. CR -HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE.

We consider an n -dimensional proper CR -hypersurface M of a complex projective space $\bar{M}$. Then it follows that $\dim \bar{D} = 1$. Now assume that M is minimal and the holomorphic distribution D is integrable. If $(\bar{e}_i), i = 1, \dots, 2p$ is an orthonormal basis for D , where $2p = \dim \bar{D}$, then $\sum_{i=1}^{2p} h(e_i, e_i) = 0$. Since M is minimal we get $h(\xi, \xi) = 0$ for ξ a unit vector in $\bar{D}$. Note that $\nabla_X \xi \in D$. Then using the equation $\bar{\nabla}_X J\xi = J\bar{\nabla}_X \xi$ and equations (2.1) and (2.2) we have for $X \in D$

$$\nabla_X \xi = JAX - h(X, \xi) \quad (3.1)$$

Also the equation $\bar{\nabla}_\xi J\xi = J\bar{\nabla}_\xi \xi$ with $h(\xi, \xi) = 0$ and equations (2.1) and (2.2) yields

$$\nabla_\xi \xi = JA\xi \quad (3.2)$$

Let $(e_i), i = 1, \dots, n$ be an orthonormal basis for M , where $e_i = \bar{e}_i$ for $i = 1, \dots, 2p$ and $e_n = \xi$. $n = 2p + 1$. Since A is symmetric and J is skew symmetric we get

$$g(JAe_i, e_i) = -g(JAJe_i, Je_i). \quad (3.3)$$

Then using (3.1), (3.2), and (3.3) we compute

$$\operatorname{div} \xi = \sum_{i=1}^n g(\nabla e_i \xi, e_i) = \sum_{i=1}^{2p} g(\nabla e_i \xi, e_i) + \sum_{i=1}^p \{g(JAe_i, e_i) + g(JAJe_i, Je_i)\} = 0 \quad (3.4)$$

For any vector field X on M we have [5]

$$\operatorname{div}(\nabla_X X) - \operatorname{div}(\operatorname{div} X)X = S(X, X) + \frac{1}{2}|L_X g|^2 - |\nabla X|^2 - (\operatorname{div} X)^2 \quad (3.5)$$

where S is the Ricci tensor and $L_X g$ is the Lie differentiation with respect to a vector field X , defined by

$$(L_X g)(Y, Z) = g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$$

Using (3.4) in (3.5) with $X = \xi$ we get

$$\operatorname{div}(\nabla_\xi \xi) = S(\xi, \xi) + \frac{1}{2}|L_\xi g|^2 - |\nabla \xi|^2 \quad (3.6)$$

From Gauss equation (2.4) and the fact that $h(\xi, \xi) = 0$ we have

$$\begin{aligned} S(\xi, \xi) &= (n-1)g(\xi, \xi) - \sum_{i=1}^n g(h(e_i, \xi), h(e_i, \xi)) = (n-1) - \sum_{i=1}^n g(h(e_i, \xi), J\xi)g(h(e_i, \xi), J\xi) \\ &= (n-1) - \sum_{i=1}^n g(A\xi, e_i)g(A\xi, e_i) = (n-1) - g(A\xi, A\xi) = (n-1) - g(A^2\xi, \xi) \end{aligned} \quad (3.7)$$

Using (3.1) and (3.2) we also have

$$\begin{aligned} |\nabla \xi|^2 &= \sum_i g(\nabla_{e_i} \xi, \nabla_{e_i} \xi) = \sum_{i,j} g(\nabla_{e_i} \xi, e_j)g(\nabla_{e_i} \xi, e_j) = \sum_{i,j} g(JAe_i, e_j)g(JAe_i, e_j) \\ &= \sum_i g(JAe_i, JAe_i) - \sum_i g(JAe_i, J\xi)g(JAe_i, J\xi) = \operatorname{trace} A^2 - \sum_i g(A\xi, e_i)g(A\xi, e_i) \\ &= \operatorname{trace} A^2 - g(A\xi, A\xi) = \operatorname{trace} A^2 - g(A^2\xi, \xi) \end{aligned} \quad (3.8)$$

From (3.6), (3.7), and (3.8) we obtain

$$\operatorname{div}(\nabla_\xi \xi) = (n-1) - \operatorname{trace} A^2 + \frac{1}{2}|L_\xi g|^2 \quad (3.9)$$

PROOF. Using equation (3.9) and the assumption that M is compact we have

$$2 \int_M [(n-1) - \operatorname{tr} A^2] dv = - \int_M |L_\xi g|^2 dv \quad (3.10)$$

From the hypothesis of Theorem and equation (3.10), we have $|L_\xi g| = 0$. Hence

$$0 = (L_\xi g)(JX, \xi) = g(\nabla_{JX} \xi, \xi) + g(\nabla_\xi JX, JX) = g(\nabla_\xi \xi, JX)$$

Using equation (3.2) in the above equation we get $h(X, \xi) = 0$ i.e., M is mixed foliate. Since the holomorphic sectional curvature c of the complex projective space $\bar{M}$ equals 4, then by theorem (2) M cannot be proper mixed foliate. Therefore M is either totally real or holomorphic. But since $\dim \bar{M} = 1$, M cannot be holomorphic. Therefore M is totally real. Since M is a hypersurface this implies that $\dim M = 1$ and $\dim \bar{M} = 2$. Now using the assumption that $\operatorname{tr} A^2 \leq n-1$ and $\dim M = 1$ we have $\operatorname{tr} A^2 = 0$ i.e., M is totally geodesic. Since $\dim \bar{M} = 2$ i.e., $\bar{M}$ is S^2 ($\equiv CP$), then M totally geodesic implies that M is a great circle S^1 on S^2 .

NOTE: It has been pointed out to us that the result in this theorem might be in conflict with Proposition 2.3 of Maeda, Y., "On real hypersurfaces of a complex projective space," J. Math. Soc. Japan, Vol. 28, No. 3.3 (1976), 529-540. We could not detect any mistakes in our proof, but we shall investigate this point later.

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